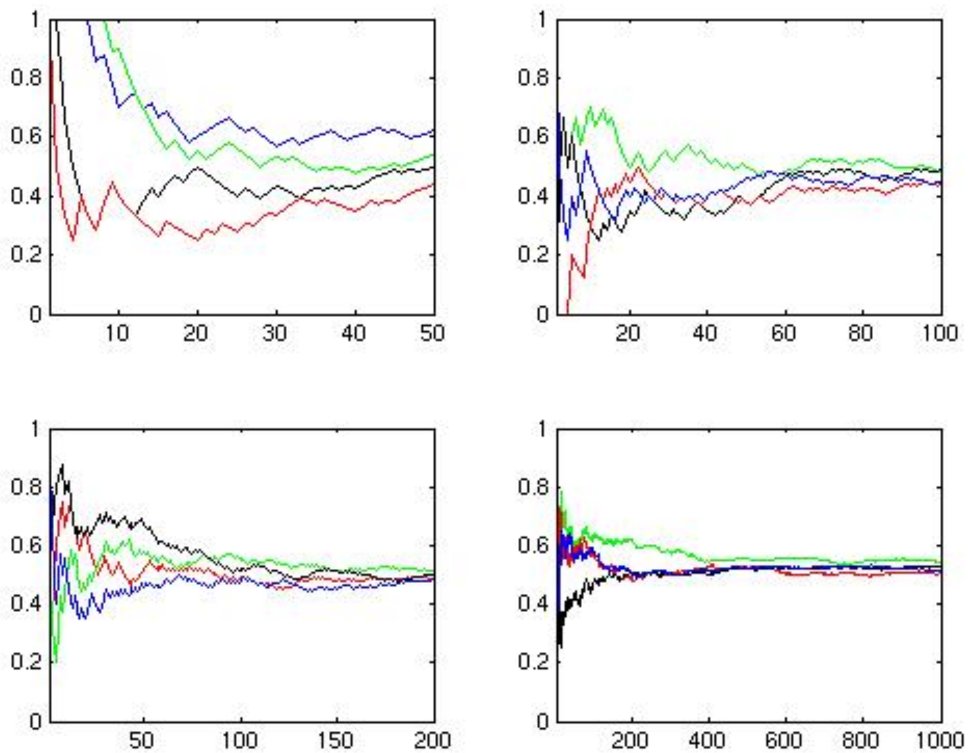
S.R.S Varadhan

by Professor Tom Louis Lindstrøm

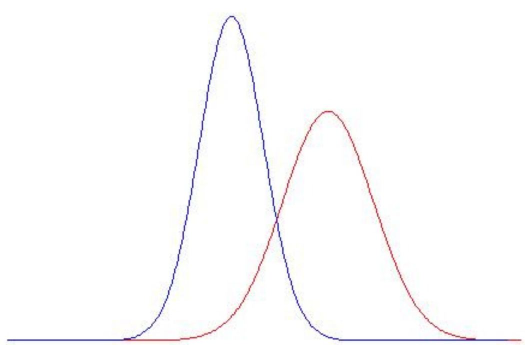
Srinivasa S. R. Varadhan was born in Madras (Chennai), India in 1940. He got his B. Sc. from Presidency College in 1959 and his Ph.D. from the Indian Statistical Institute in 1963. Since 1963 he has been working at the Courant Institute of Mathematical Sciences at New York University. The Courant Institute — one the worlds leading centers of applied mathematics — is also the home institution of the 2005 Abel Laureate, Peter Lax.

Although his work is often motivated by problems in neighboring fields such as mathematical physics and partial differential equations, Varadhan is primarily a probabilist. Historically, probability theory started as an attempt to understand simple games of chance, often with betting involved. In such games there is a finite number of possible outcomes, and the task is to find the probability of each one. Although this may sound simple, these problems often require a lot ingenuity. The subject soon moved on, however, to more important and difficult issues. These problems often have to do with what happens if one repeats the same experiment over and over again, and the mathematical laws governing these repeated experiments are often called *limit laws* as they describe what happens "in the limit" as one performs the experiments more and more times. Two of these limit laws we all have some experience with.



To describe the first of these laws, let us assume that we are tossing a coin many times. If the coin is fair, we would expect the proportion of "heads" to stabilize around 1/2 as we toss the coin more and more times. This is an instance of *the Law of Large of Numbers* which says that if we repeat the experiment more and more times, the proportion of heads will (with probability one) go to exactly 1/2. The figures above illustrate what happens. In each case we have performed four sequences of coin tosses and computed the proportion of “heads”. The figure in the upper, left hand corner shows the results when we toss the coin 50 times. The other figures show the results when we toss the coin 100 times, 200 times and 1000 times, respectively. We see that the convergence is rather slow; even when we toss the coin 1000 times, the results are clearly separable.

To describe the other limit law that we all have some experience with, let us assume that we are measuring the heights of Norwegian (male) soldiers. If we make a diagram of how many soldiers there are of each height (i.e. how many are 175 cm tall, how many are 176 cm etc.), we soon see that this diagram takes on a bell shaped form, and as we add the heights of more and more soldiers, the curve becomes more and more regular. The curve is symmetric around its center, and the center point is the average height (around 180 cm for Norwegian male soldiers). If we did the same experiment with female soldiers, we would get the same kind of bell shaped curve, but with a different center (as woman are in average not as tall as men) and a slightly different width. What we are seeing here are consequences of the *Central Limit Theorem* which basically says that statistical properties which depend on a lot of small, independent factors have a bell shaped distribution. These bell shaped distributions are called *normal distributions*, and different normal distributions are described by two numbers (the mean and the variance) — one telling us where the center is and the other telling us how wide and flat the curve is. The figure below shows two normal distributions with different centers and different widths, i.e. different means and different variances.



Why are these limit laws important? They are important because in most practical situations one is interested in large collections of statistical data. If you are running a car insurance company, you are not interested in each individual car, but you are interested in how many accidents (and what kinds of accidents) *all* the cars you insure will be in involved in. If you are building an oil drilling platform in the North Sea, and you worry about the impact of the ocean waves on the construction, you don’t worry about the impact of each individual wave, but of the collective impact of all of them. If you are building a telephone network and worry about the capacity, you are not interested in each individual customer, but you worry about the probability that too many of them will pick up the phone at the same time during peak hours.

If you look at the mathematical problems suggested by the examples above, you will find that many of them can be solved using the Law of Large Numbers and the Central Limit Theorem. But not all! An interesting question they can not tackle, is the question of "large deviations". To explain this problem, let us go back to coin tossing. If we flip a coin many times, we expect the proportion of "heads" to be around 1/2. But this *need* not happen — even if you flip the coin a thousand times, there is a small (extremely small!) probability that the coin will show "heads" every time. There is a larger — but still extremely small — probability that the proportion of "heads" will be (say) 3/4 instead of 1/2. The art of *large deviations* is to calculate the probability of such rare events.

Large deviations were first studied by the great Swedish statistician and insurance mathematician Harald Cramér (1893-1985) in the late 1930s. It is easy to see why the problem would attract the attention of somebody interested in insurance mathematics. The premium you have to pay for your car insurance, is based on the statistics from previous years — the company needs to collect enough money to cover the claims of unfortunate drivers. But what if this year happens to be an extremely bad year — that for some unforeseen reason (perhaps just bad luck) much more cars crash than previous years? If the company has to pay out more money than it has got, it is obviously in trouble!

There is no way one can totally avoid this problem — if the company made the premium so high that it covered the unlikely (but still possible) case that *all* cars crashed, the insurance would be so expensive that nobody would buy it! Now, a bad year is a large deviation, and what the company needs to do, is to compute the probability of these large deviations of various sizes, and try to find a reasonable level of risk. There are similar problems in our other examples above — if you build an oil rig in the North Sea, you have to worry about the probability of extremely large and extremely rare waves ("the hundred year wave"), and if you set up a telephone network, you need to know how likely it is that it will occasionally break down due to overload. It may be worthwhile to invest a little more in increased capacity rather than have to face angry customers every now and then!

One of Varadhan's great contributions is to turn the technique of large deviations into a very strong and versatile tool with applications in many areas of mathematics and related sciences (some of this work was done in collaboration with his colleague at the Courant Institute, Monroe Donsker). Varadhan's *Large Deviation Principle* succinctly sums up what is needed to apply the technique successfully, and it covers a surprising number of seemingly different situations. The theory is a tour de force of many areas of mathematics; it combines probability theory with convex analysis, nonlinear programming, functional analysis and partial differential equations. It turns out that the theory of large deviations is much more subtle than the theory of classical limit laws such as the Law of Large Numbers and the Central Limit Theorem. In these limit laws, the important thing is that the same kind of event is counted again and again; the nature of each individual event is of little importance (or, more correctly, what is important is easily summarized in two numbers, the mean and the variance). In large deviations, the nature of the individual events is of the outmost importance — different kinds of events give rise to quite different probabilities for large deviations. An insurance company which wants to compute precise estimates for large deviations hence needs to know more about car accidents than just how often they happen and how much they cost on average.

In addition to the examples I have already referred to, I would like to add the many applications that large deviation theory has found in mathematical physics. Many physical theories are statistical in nature. If, for instance, you want to describe the air in this room or the water in a flowing river, you cannot possible describe the motion of each individual molecule or particle involved. Instead you describe the *statistical behavior* of the totality of particles in terms of macroscopic quantities such as pressure and flux. The laws and equations you then get are *not* probabilistic, they just describe the average or expected behavior of the gas or the fluid. You may think of this as a much more complicated version of our coin-tossing experiment — in that experiment, the average behavior of our probabilistic experiment was summed up in the simple number 50%; in the present case, the total probabilistic behavior of the particles is summed up in the laws of thermodynamics and hydrodynamics! But as in the coin-tossing game, there are fluctuations also in this situation — perhaps there is a very small probability that all the air in this room will suddenly concentrate on this end and leave you suffocated at the other end!

In fact, these problems are even more complicated than what I have described so far since the behavior of the particles is not really random at all — they move according to the fundamental laws of physics, and their behavior only *looks* random because they collide and interact all the time. The derivation of the equations of hydrodynamics and thermodynamics from first principles is thus a two stage process — first to derive the statistical behavior of the particles from the laws of physics and then to deduce macroscopic laws from this statistical description. With collaborators, Varadhan has done impressive work in this area, often using large deviations as a tool.

Let us go back to the coin tossing experiment to take a look at another aspect of Varadhan's work. We can think of coin tossing as a very simple gambling game — each time we toss the coin and "heads" come up, I win and you have to pay me a dollar, but each time "tails" come up, you win and I have to pay you a dollar. This game is "fair" in an obvious sense — in average, I cannot expect to win anything and neither can you! Let us change the game slightly and throw a die instead of a coin. This time I get 5 dollars from you each time we throw a "six", but you get one dollar from me each time one of the other sides comes up. The game is still "fair" in the sense I described above — I get five times as much as you each time I win, but you can expect to win five times as often, and neither of us can expect to win anything in the long run. Games which are fair in this sense are called *martingales,* and this notion can be generalized to much more general contexts. Over the last fifty years, it has become clear that martingales are extremely useful tools in the study of random phenomena. In the 1970s, Varadhan and D.W. Stroock wrote an impressive series of papers on so called "martingale problems" culminating in their book “Multidimensional Diffusion Processes” of 1979. Their approach unified, simplified and extended the previous results in the area substantially. The basic idea is that instead of looking for solutions to quite complicated problems of mathematical analysis, “all” one has to look for is a probability distribution which turns certain processes into martingales.

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I have mentioned some the most important of Varadhan’s contributions to mathematics, but there are many more. His is a prolific scientist with deep insights and an impressive array of technical tools, and he is very highly regarded and esteemed in the probability community. This does not only have to do with his results, but also his style — listening to a lecture by Varadhan, one is not only exposed to the best and most recent results in the subject, but one is also introduced to a way of thinking. His talks always emphasize the basic ideas, the challenges, the obstacles, and the delicate balance between the desirable and the possible which one has to strike to produce top class mathematical results. S. R. S. Varadhan is certainly a worthy winner of the Abel Prize!