Abel Prize 2008

A historical view on the Theory of Groups

This year's Abel Prize is not only a prize for John G. Thompson and Jacques Tits. It may also be considered as a recognition of the work performed by hundreds of group theorists through several generations. We shall give a sketchy overview of the history of group theory, its milestones and its founders.

The development of group theory as an area of research was initiated about 200 years ago, although the term ***abstract group*** is somewhat younger. It was mathematicians like ***Arthur Cayley, Camille Jordan*** and ***Walther von Dyck*** who towards the end of the nineteenth century gave the formal definition used today.

It took a long time to fix the term 'group'. ***Augustin Louis Cauchy*** used the far more complicated term *conjugate systems of substitutions*, but a paper of Jordan in 1863 seems to have settled the term we use today.

By the time Jordan's paper appeared, 30 years had gone by since the man who was the first to use the term group in this context, ***Evariste Galois***, was killed in a duel, at the age of 20. Galois was politically active during the French revolution in 1830 and the duel may not have been what it seemed to be, namely a fight between young French men for the heart of a pretty woman. Rather, it may well have been put on stage by the political opponents of Galois in order to get rid of a troublesome opponent. Nevertheless, the myth says that the young Galois spent the night before the duel writing down his mathematical testament. In this seminal work, rather sketchy written, and which forms the foundation of what was later to be called Galois theory, Galois studied solutions of polynomial equations and permutations of the solutions. This turned out to be rather fruitful in deciding whether the equations could be solved by radicals. In this way, Galois generalised the earlier work of ***Niels Henrik Abel,*** where Abel proved that quintic equations cannot, in general, be solved by radicals. Galois introduced the term 'group' of permutations, without any formal definition, nor stating any general properties. To him it was only a collection of permutations. But he made one important remark:

> *... if in such a group one has the substitutions S and T, then one also has the substitution ST.*

Thus Galois was aware of the importance of some structure in his group, namely that the composition (or product) of two elements in a group should also be an element in the group. In modern terms this is expressed by stating that a group is closed under a binary operation. After Galois´ too early death, several prominent mathematicians continued to work on the group concept. Important contributions were made by Cauchy and Jordan, but the first one to try to give a formal definition was Cayley in 1854. However, it was not until a paper of von Dyck appeared in 1882 that the formal definition as we know it today was born.

Permutations and solutions of polynomial equations are not the only ways to consider group theory. Equally important is the geometric point of view. The collection of symmetries of a geometric object form a group structure in a natural way. As an example of a symmetry group we can examine transformations of a circle. Generated by rotations and reflections they look very much like the symmetries of a square. The difference is the size; whereas the square has finitely many symmetries, the circle has an infinite number of symmetries.

Yet another point of view was due to the Norwegian mathematician ***Sophus Lie***, in particular his work dated 1884. Motivated by an idea of his fellow citizen Abel of understanding equations and their solutions by studying permutations of the solutions, Lie paid his attention to differential equations and asked a similar question. Like Abel, Lie focused on the fundamental connections between geometry and algebra and his transformation groups are now known as Lie groups. Neither Abel nor Lie invented group theory, still there are important classes of groups carrying their names, namely 'abelian groups' and 'Lie groups'.

Further to this, yet another Norwegian mathematician has his name connected to a well known class of groups: During the fall semester in 1862, the University of Christiania (now Oslo) had to find a substitute to teach courses at a graduate level. Professor ***Ole Jacob Broch*** was elected as a member of the Parliament and had no time for teaching, ans so the young mathematician ***Ludwig Sylow*** was chosen for this purpose. Sylow had graduated from the University some years earlier, but due to lack of faculty positions he served as a teacher at the *Fredrikshalds Lærde og Realskole,* in a small town now called Halden. During his two months in Christiania he lectured on Galois theory, the first time this topic was lectured in Norway. One of his lectures was on Cauchy's theorem on the existence of subgroups of prime order, for primes dividing the order of the group. At the end of the lecture Sylow asked the audience about a generalisation of this theorem: Is the theorem also valid for maximal powers of the prime? Neither he, nor the audience, including the young Sophus Lie, were able to answer this question at the time. However, 10 years later he could proudly publish the solution: The answer was 'yes', and there was more to it: Sylow was able to give the number of such groups, later known as 'Sylow subgroups', and he also showed that these subgroups are closely related. What Sylow didn´t know in 1862 nor in 1872 was that his 10-page paper was to have astonishing influence on a huge project in group theory, which was to be fulfilled more than a century later; the classification of all finite simple groups, including the discovery of the 26 sporadic groups, with the *Monster group* as the final member. The Monster is the biggest finite simple group and has order

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It was at the International Congress of Mathematicians in Amsterdam in 1954 that ***Richard Brauer*** presented his vision of classifying all finite simple groups. Almost 30 years later, ***Daniel Gorenstein***, who overviewed it all, declared the project concluded. Hundreds of mathematicians had at that time published thousands of pages on what became a joint proof of the classification of finite simple groups. Amongst all the skilled group theorists, a young mathematician named ***John G. Thompson*** gave the far most important contribution, the so-called *Feit-Thompson theorem*, which he proved together with ***Walter Feit*** in 1962/63.

The classification of all finite simple groups did not put an end to group theory, but rather a new start. From the first time Galois used the term *group* to the conceptual canonisation of an abstract group appeared, half a century had passed. Then it took another whole century to classify all the atoms of the theory!