Abel Prize 2009

The contributions of Mikhail Gromov

The citation of the Abel commitée emphasises three particular areas of achievements made by Mikhail Gromov: Global Riemannian geometry, symplectic geometry and groups of polynomial growth.

Riemannian Geometry
From the citation: *Gromov played a decisive role in the creation of modern global Riemannian geometry. His solutions of important problems in global geometry relied on new general concepts, such as convergence of Riemannian manifolds and a compactness principle, which now bear his name.*

Georg Friedrich Bernhard Riemann (1826-1866)

Riemannian geometry is named after the German mathematician Georg Friedrich Bernhard Riemann (1826-1866). Riemann was a student of the great Carl Friedrich Gauss (1777-1855).
In 1853, Gauss asked Riemann to prepare a Habilitationsschrift on the foundations of geometry. Over many months, Riemann developed his theory of higher dimensions. When he finally delivered his lecture at Göttingen in 1854, the mathematical audience received it with enthusiasm, and it became one of the most important works in geometry. It was entitled *Über die Hypothesen welche der Geometrie zu Grunde liegen* (loosely: "*On the foundations of geometry*"; more precisely, "*On the hypotheses which underlie geometry*"), and was published in 1868.

The subject founded by this work is Riemannian geometry. Riemannian geometry is the branch of differential geometry that studies so-called Riemannian manifolds. A Riemannian manifold is a smooth manifold with a Riemannian metric, i.e. with an inner product on the tangent space at each point which varies smoothly from point to point. This gives local notions of angle, length of curves, surface area, and volume.

In the 1980´s Gromov introduced what is now called the Gromov-Hausdorff distance between two abstract metric spaces. The distance is measured by embedding the two spaces into a third bigger space. Hausdorff had suggested a way of measuring the distance between the two spaces. Gromov proved two fundamental results for this construction, a precompactness theorem and a convergence theorem.

Symplectic geometry
From the citation: *Gromov is one of the founders of the field of symplectic geometry. Holomorphic curves were known to be an important tool in the geometry of complex manifolds. However, the environment of integrable complex structures was too rigid. In a famous paper in 1985, Gromov extended the concept of holomorphic curves to J-holomorphic curves on symplectic manifolds. This led to the theory of Gromov-Witten invariants, which is now an extremely active subject linked to modern quantum field theory. It also led to the creation of symplectic topology and gradually penetrated and transformed many other areas of mathematics.*
The term "*symplectic*" is a calque of "complex", by Hermann Weyl; previously, the "symplectic group" had been called the "line complex group".

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Complex comes from the Latin com-plexus, meaning ”braided together” (co- + plexus), while symplectic comes from the corresponding Greek sym-plektos (συμπλεκτικός); in both cases the suffix comes from the Indo-European root *plek-.This naming reflects the deep connections between complex and symplectic structures.

Symplectic geometry is a branch of differential geometry and differential topology which studies symplectic manifolds; that is, differentiable manifolds equipped with a closed, nondegenerate 2-form. Symplectic geometry has its origins in the Hamiltonian formalism for classical mechanics where the phase space of certain classical systems takes on the structure of a symplectic manifold. Symplectic geometry and Riemannian geometry have a number of similarities, but also distinctions between them. Unlike in the Riemannian case, symplectic manifolds have no local invariants such as curvature. Furthermore it is that not every differentiable manifold that admit a symplectic form; there are certain topological restrictions. For example, every symplectic manifold is evendimensional and orientable.

Gromov used the existence of almost complex structures on symplectic manifolds to develop a theory of pseudoholomorphic curves, which has led to a number of advancements in symplectic topology, including a class of symplectic invariants now known as Gromov-Witten invariants. These invariants also play a key role in string theory.

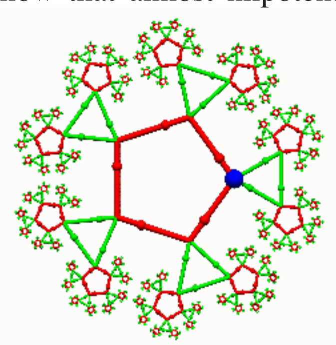
Groups of polynomial growth

From the citation: *Gromov's work on groups of polynomial growth introduced ideas that forever changed the way in which a discrete infinite group is viewed. He discovered the geometry of discrete groups and solved several outstanding problems. His geometrical approach rendered complicated combinatorial arguments much more natural and powerful.*

In group theory, the growth rate of a group with respect to a symmetric generating set describes the size of balls in the groupwhen viewed as a geometrical object. Every element in the group can be written as a product of generators, and the growth rate counts the number of elements that can be written as a product of length n for increasing values of n. Gromov proved the following theorem:

Theorem (Gromov 1981).
A finitely generated group G has polynomial growth if and only if it is virtually nilpotent.

The theorem says that a finitely generated group G has polynomial growth if and only if G is ”almost” nilpotent. Having polynomial growth is a geometric condition on the Cayley graph, whereas being almost nilpotent is an entirely algebraic property. It is not too hard to show that almost nilpotent groups have polynomial growth; the hard part of the theorem is the converse, and Gromov introduced several new geometric notions in order to convert the geometric information into an algebraic conclusion. These ideas have become the basis for many current strategies for approaching problems in the field of geometric group theory.



Cayley graph of $\mathbf{Z}_3$x$\mathbf{Z}_5$