Memoirs of My Research on Stochastic Analysis

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It is with great honor that I learned of the 2005 Oslo Symposium on Stochastic Analysis and Applications, which is devoted to my work and its further developments. I would like to thank the symposium organizers for their tireless efforts in organizing this successful symposium and for providing me with the opportunity to present some memoirs of my research on stochastic analysis, which, I hope, will be of some interest to the participants.

My doctoral thesis published in 1942 [1] was on a decomposition of the sample path of the continuous time stochastic process with independent increments, now called the Lévy-Itô decomposition of the Lévy process. In the 1942 article written in Japanese [2] and the extended 1951 version that appeared in the Memoirs of the American Mathematical Society [3], I succeeded in unifying Lévy's view on stochastic processes and Kolmogorov's approach to Markov processes and created the theory of stochastic differential equations and the related stochastic calculus. As is beautifully presented in a recent book by Daniel Stroock [11], *Markov Processes from K.Itô's Perspective,* my conception behind those works was to take, in a certain sense, a Lévy process as a tangent to the Markov process. The above mentioned papers are reprinted in *Kiyosi Itô Selected Papers* edited by Stroock and Varadhan [10], and the editors' introduction and my own foreword explain in some detail the circumstances leading to their development.

From 1954 to 1956, I was a Fellow at the Institute for Advanced Study at Princeton University, where Salomon Bochner and William Feller, both great mathematicians, were among the faculty members. In the preceding year, while still at Kyoto University, I had written a paper on stationary random distributions [4], using a Laurent Schwartz's extension of Bochner's theorem to a positive definite distribution representing it by a slowly increasing measure. As I learned from Bochner in Princeton, this had essentially already been obtained by Bochner himself by other means.

Feller had just finished his works on the most general one dimensional diffusion process, especially representing its local generator as

$$\mathcal{G} = \frac{d}{dm}\frac{d}{ds}$$

by means of a canonical scale function s and a speed measure m. I learned about these from Henry McKean, a graduate student of Feller, while I explained my previous work to McKean. There was once an occasion when McKean tried to explain to Feller my work on the stochastic differential equations along with the above mentioned idea of tangent. It seemed to me that Feller did not fully understand its significance, but when I explained Lévy's local time to Feller, he immediately appreciated its relevance to the study of the one dimensional diffusion. Indeed, Feller later gave us a conjecture that the Brownian motion on $[0, \infty)$ with an elastic boundary condition could be constructed from the reflecting barrier Brownian motion by killing its local time $\mathbf{t}(t, 0)$ at the origin by an independent exponentially distributed random time, which was eventually substantiated in my joint paper with McKean [7] published in 1963 in the Illinois Journal of Mathematics.

After my return to Kyoto from Princeton, McKean visited Kyoto in 1957-1958, and our intensive collaboration continued until our joint book *Diffusion Processes and Their Sample Paths* appeared from Springer in 1965 [8]. This coincides with the period when Dynkin and Hunt formulated the general theory of strong Markov processes along with their transformations by additive functionals and the associated probabilistic potential theory. The Kyoto probability seminars attracted many young probabilists in Japan; S. Watanabe, H. Kunita and M. Fukushima were among my graduate students. The primary concern of the seminar participants including myself was to fully understand the success of the study of one dimensional diffusions and to look for its significant extensions to more general Markov processes. Let me mention some of the later developments of a different character that grew out of this exciting seminar atmosphere.

A popular saying by Feller goes as follows: A one dimensional diffusion traveler X_t makes a trip in accordance with the road map indicated by the scale function s and with the speed indicated by the measure m appearing in the generator $\mathcal{G}$ of X_t. This was substantiated in my joint book with McKean in the following fashion. Given a one dimensional standard Brownian motion X_t which corresponds to $ds = dx,\ dm = 2dx$, consider its local time $\mathbf{t}(t, x)$ at $x \in R^1$ and the additive functional defined by

$$A_t = \int_{R^1} \mathbf{t}(t, x) m(dx).$$

Then the time changed process X_{τ_t} by means of the inverse τ_t of A_t turns out to be the diffusion governed by the generator $\frac{d^2}{dmdx}$.

Observe that the transition function of the one dimensional diffusion is symmetric with respect to the speed measure m and the associated Dirichlet form

$$\mathcal{E}(u,v) = -\int_{R^1} u \cdot \mathcal{G}v(x)dm(x) = \int_{R^1} \frac{du}{ds}\frac{dv}{ds}ds$$

is expressed only by the scale s, being separated from the symmetrizing measure m. Hence we are tempted to conjecture that the 0-order Dirichlet form $\mathcal{E}$ indicates the road map for the associated Markov process X_t and is invariant under the change of the symmetrizing measures m corresponding to the random time changes by means of the positive continuous additive functionals of X_t. The notion of the Dirichlet form was introduced by Beurling and Deny as a function space framework of an axiomatic potential theory in 1959, where already the road map was clearly indicated in analytical terms (the Beurling-Deny formula of the form) but the role of the symmetrizing measure m was much less clear. Being led by the above-mentioned picture of the one dimensional diffusion path, the conjecture has been affirmatively resolved in later works by Fukushima and others (see the 1994 book by Fukushima, Takeda and Oshima *Dirichlet Forms and Symmetric Markov Processes,* [12]).

In 1965, M. Motoo and S. Watanabe wrote a paper [13] in which they made a profound analysis of the structure of the space of square integrable martingale additive functionals of a Hunt Markov process. In the meantime, the Doob-Meyer decomposition theorem of submartingales was completed by P.A. Meyer. These two works merged into a paper by H. Kunita and S. Watanabe which appeared in the Nagoya Mathematical Journal in 1967 [14] and a series of papers by P.A.Meyer in the Strasbourg Seminar Notes in 1967 [15], where the stochastic integral was defined for a general semi-martingale, and the stochastic calculus I initiated in 1942 and 1951 was revived in a new general context. Since then, various researchers including myself also became more concerned about the stochastic calculus and stochastic differential equations.

My joint paper [7] with McKean in 1963 gave a probabilistic construction of the Brownian motion on $[0,\infty)$ subjected to the most general boundary condition whose analytic study had been established by Feller under some restrictions. Our methods involved the probabilistic idea originated in Lévy about the local time and excursions away from 0. In 1970, the idea was extended in my paper in the Proceedings of the Sixth Berkeley Symposium [9], where I considered a general standard Markov process X_t for which a specific one point a is regular for itself. A Poisson point process taking values in the space U of excursions around point a was then associated, and its characteristic measure (a σ-finite measure on U) together with the stopped process obtained from X_t by the hitting time of a was shown to uniquely determine the law of the given process X_t. This approach may be considered as an infinite dimensional analogue to a part of the decomposition of the Lévy process I studied in 1942, and may have revealed a new aspect in the study of Markov processes.

The one dimensional diffusion theory is still important as a basic prototype of Markov processes. Besides my joint book [8] with McKean, I also gave a comprehensive account of the Feller generator as a generalized second order differential operator in section 6 of my *Lectures on Stochasitc Processes* at the Tata Institute of Fundamental Research, Bombay, 1960 [6]. The second part of my book *Stochastic Processes* [5] written in Japanese and published in 1957 contains a detailed description of the Feller generator and, in addition, of the boundary behaviors of the solutions of the associated homogeneous equation

$$(\lambda - \mathcal{G})u = 0, \qquad \lambda > 0,$$

in an analytical way together with their probabilistic implications. While the first part of this book [5] was translated into Russian by A.D. Wentzell, the second part has been appreciated exclusively by Japanese mathematicians. In 1959 Shizuo Kakutani at Yale University, noting the importance of my description of the one dimensional diffusions, advised Yuji Ito, then one of his graduate students, to produce a translation of the second part into English, which was then distributed among a limited circle of mathematicians around Yale University as a typewritten mimeograph. I am very glad to hear that a full English translation of the book [5] by Yuji Ito is now being prepared for publication by the American Mathematical Society under the title *Essentials of Stochastic Processes.*

Finally, let me extend my deepest gratitude to the symposium organizers and participants for honoring my 90th birthday with your work on stochastic analysis. I also wish to thank you again for allowing me to present these memoirs to you here, and I very much look forward to studying all the papers presented at this symposium.

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