XIII.

Sur quelques intégrales définies.

On a vu précédemment que

$$\int_0^{\frac{\pi}{2}} \frac{(\cos\varphi)^n.\cos n\varphi.d\varphi}{x^2.\sin^2\varphi+\alpha^2.\cos^2\varphi}=\frac{\pi}{2}\cdot\frac{x^{n-1}}{\alpha(x+\alpha)^n};$$

or $$(\cos\varphi)^n=1+n.\log\cos\varphi+\frac{n^2}{2}(\log\cos\varphi)^2+\ldots$$

$$\cos n\varphi=1-\frac{n^2}{2}\cdot\varphi^2+\frac{n^4}{2.3.4}\cdot\varphi^4+\ldots$$

donc

$$(\cos\varphi)^n.\cos n\varphi=1+n.\log\cos\varphi+\frac{n^2}{2}((\log\cos\varphi)^2-\varphi^2)+\frac{n^3}{2.3}((\log\cos\varphi)^3-3\varphi^2(\log\cos\varphi))$$
$$\ldots+\frac{n^m}{\Gamma(m+1)}\cdot A_m$$

où on a, en faisant pour abréger $\log\cos\varphi=t$:

$$\frac{A_m}{\Gamma(m+1)}=\frac{t^m}{\Gamma(m+1)}-\frac{t^{m-2}.\varphi^2}{\Gamma(3).\Gamma(m-1)}+\frac{t^{m-4}.\varphi^4}{\Gamma(5).\Gamma(m-3)}+\frac{t^{m-6}.\varphi^6}{\Gamma(7)\Gamma(m-5)}+\ldots$$

or $$\frac{x^n}{(x+\alpha)^n}=1+n.\log\frac{x}{x+\alpha}+\frac{n^2}{2}\cdot\left(\log\frac{x}{x+\alpha}\right)^2+\ldots$$

donc on aura

$$\frac{\pi}{2}\cdot\frac{1}{x\alpha}\cdot\left(\log\frac{x}{x+\alpha}\right)^m=\int_0^{\frac{\pi}{2}}\frac{A_m.d\varphi}{x^2.\sin^2\varphi+\alpha^2.\cos^2\varphi}.$$

Ainsi on aura:

$$\frac{\pi}{2}\cdot\frac{1}{x\alpha}=\int_0^{\frac{\pi}{2}}\frac{d\varphi}{x^2.\sin^2\varphi+\alpha^2.\cos^2\varphi}$$

$$\frac{\pi}{2}\cdot\frac{1}{x\alpha}\log\frac{x}{x+\alpha}=\int_0^{\frac{\pi}{2}}\frac{\log\cos\varphi.d\varphi}{x^2.\sin^2\varphi+\alpha^2.\cos^2\varphi}$$

$$\frac{\pi}{2}\cdot\frac{1}{x\alpha}\cdot\left(\log\frac{x}{x+\alpha}\right)^2=\int_0^{\frac{\pi}{2}}\frac{[(\log\cos\varphi)^2-\varphi^2].d\varphi}{x^2.\sin^2\varphi+\alpha^2.\cos^2\varphi}.$$

En faisant $x = \alpha$ on aura

$$\frac{\pi}{2} \cdot (\log \tfrac{1}{2})^m = \int_0^{\frac{\pi}{2}} A_m \,.\, d\varphi;$$

par exemple $$\frac{\pi}{2} \cdot \log \tfrac{1}{2} = \int_0^{\frac{\pi}{2}} d\varphi \,.\, \log \cos \varphi.$$

Soit $\cos \varphi = y$, on aura $d\varphi = -\dfrac{dy}{\sqrt{(1-y^2)}}$, donc

$$\frac{\pi}{2} \log 2 = \int_1^0 \frac{\log y \,.\, dy}{\sqrt{(1-y^2)}}.$$

En effet on a

$$\int_0^1 \frac{x^{p-1} dx \,.\, \log\left(\frac{1}{x}\right)}{\sqrt[n]{(1-x^n)^{n-q}}} = \int_0^1 \frac{x^{p-1} dx}{\sqrt[n]{(1-x^n)^{n-q}}} \cdot \int_0^1 \frac{(x^{p-1} - x^{p+q-1} dx}{1-x^n};$$

donc $$\int_0^1 \frac{\log\left(\frac{1}{y}\right) . \, dy}{\sqrt{(1-y^2)}} = \int_0^1 \frac{dy}{\sqrt{(1-y^2)}} \cdot \int_0^1 \left(\frac{1-y}{1-y^2}\right) dy:$$

or $$\int_0^1 \frac{dy}{\sqrt{(1-y^2)}} = \frac{\pi}{2} \text{ et } \int_0^1 \left(\frac{1-y}{1-y^2}\right) dy = \log 2.$$

On a $$\int_0^{\frac{1}{0}} \frac{dt}{1+t^2} \cdot \frac{\varphi(x+\alpha t\sqrt{-1}) + \varphi(x-\alpha t\sqrt{-1})}{2} = \frac{\pi}{2} \varphi(x+\alpha).$$

Soit $\varphi x = (\log x)^n$, on aura

$$\int_0^{\frac{1}{0}} \frac{dt}{1+t^2} \cdot \frac{[\log(x+\alpha t\sqrt{-1})]^n + [\log(x-\alpha t\sqrt{-1})]^n}{2} = \frac{\pi}{2} . (\log(x+\alpha))^n$$

or on a:

$$\log(x+\alpha t\sqrt{-1}) = \log x + \log\left(1+\frac{\alpha}{x} t\sqrt{-1}\right) = \log x + \frac{\log\left(1+\frac{\alpha}{x}t\sqrt{-1}\right) - \log\left(1-\frac{\alpha}{x}t\sqrt{-1}\right)}{2}$$

$$+ \tfrac{1}{2} \log\left(1+\frac{\alpha^2}{x^2} t^2\right) = \log x + \tfrac{1}{2} \log\left(1+\frac{\alpha^2}{x^2} t^2\right) - \sqrt{-1} . \text{arc tang}\left(\frac{\alpha t}{x}\right)$$

$$= \tfrac{1}{2} \log(x^2 + \alpha^2 t^2) - \sqrt{-1} . \text{arc tang}\left(\frac{\alpha t}{x}\right).$$

Soit $\dfrac{\alpha t}{x} = \operatorname{tang} \varphi$, on aura

$$\log(x+\alpha t\sqrt{-1}) = \log x - \log \cos \varphi - \sqrt{-1} . \varphi$$

$$\frac{dt}{1+t^2} = \frac{\alpha x \,.\, d\varphi}{x^2 . \sin^2\varphi + \alpha^2 . \cos^2\varphi};$$

donc

$$\int_0^{\frac{\pi}{2}} \frac{d\varphi}{x^2.\sin^2\varphi+\alpha^2.\cos^2\varphi}\cdot\frac{\left(\log\frac{x}{\cos\varphi}-\varphi\sqrt{-1}\right)^n+\left(\log\frac{x}{\cos\varphi}+\varphi\sqrt{-1}\right)^n}{2}=\frac{\pi}{2x\alpha}(\log(x+\alpha))^n.$$

En faisant $x=\alpha=1$, on aura

$$\int_0^{\frac{\pi}{2}} d\varphi.\left[\left(\log\frac{1}{\cos\varphi}-\varphi\sqrt{-1}\right)^n+\left(\log\frac{1}{\cos\varphi}+\varphi\sqrt{-1}\right)^n\right]=\pi(\log 2)^n.$$

On a aussi en général, en faisant $t=\text{tang}\, u$:

$$\int_0^{\frac{\pi}{2}} du(\varphi(x+\alpha\,\text{tang}\, u.\sqrt{-1})+\varphi(x-\alpha.\text{tang}\, u.\sqrt{-1}))=\pi\varphi(x+\alpha);$$

donc en faisant $x=\alpha=1$, on aura:

$$\int_0^{\frac{\pi}{2}} du(\varphi(1+\sqrt{-1}\,\text{tang}\, u)+\varphi(1-\sqrt{-1}\,\text{tang}\, u))=\pi.\varphi(2).$$

Soit $\varphi x=\frac{x^m}{1+\alpha x^n}$, on aura

$$\varphi(1+\sqrt{-1}\,\text{tang}\, u)=\frac{(1+\sqrt{-1}\,\text{tg}\, u)^m}{1+\alpha(1+\sqrt{-1}\,\text{tg}\, u)^n}=\frac{(\cos mu+\sqrt{-1}\sin mu)(\cos u)^{n-m}}{(\cos u)^n+\alpha.\cos nu+\alpha\sqrt{-1}.\sin nu}$$

$$=\frac{(\cos u)^{n-m}}{[(\cos u)^n+\alpha.\cos nu]^2+\alpha^2.\sin^2 nu}.\left[(\cos u)^n.\cos mu+\alpha\cos(m-n)u\right.$$

$$\left.+\sqrt{-1}\left((\cos u)^n.\sin mu+\alpha\sin(m-n)u\right)\right];$$

on tire de là

$$\int_0^{\frac{\pi}{2}} \frac{(\cos u)^{n-m}.[\cos mu.(\cos u)^n+\alpha.\cos(n-m)u]}{(\cos u)^{2n}+2\alpha.\cos nu(\cos u)^n+\alpha^2}\,du=\frac{\pi}{2}\cdot\frac{2^m}{1+\alpha.2^n}.$$

Soit $m=0$, on aura

$$\int_0^{\frac{\pi}{2}} \frac{(\cos u)^n[(\cos u)^n+\alpha\cos nu]du}{(\cos u)^{2n}+2\alpha\cos nu.(\cos u)^n+\alpha^2}=\frac{\pi}{2}\cdot\frac{1}{1+\alpha.2^n}.$$

Soit $m=n$, on aura

$$\int_0^{\frac{\pi}{2}} \frac{\cos nu.(\cos u)^n+\alpha}{(\cos u)^{2n}+2\alpha\cos nu.(\cos u)^n+\alpha^2}\,du=\frac{\pi}{2}\cdot\frac{2^n}{1+\alpha.2^n}.$$

Si par exemple $n=1$, on aura

$$\frac{\pi}{1+2\alpha}=\int_0^{\frac{\pi}{2}} \frac{(\cos u)^2+\alpha}{(\cos u)^2(1+2\alpha)+\alpha^2}\,du=\int_0^1 \frac{y^2+\alpha}{y^2(1+2\alpha)+\alpha^2}\cdot\frac{dy}{\sqrt{(1-y^2)}}$$

Reprenons la formule

$$\frac{\pi}{2}\cdot\frac{1}{2^n}=\int_0^{\frac{\pi}{2}}(\cos\varphi)^n.\cos n\varphi.d\varphi.$$

Soit $n = \frac{m}{n}$, on aura

$$\frac{\pi}{2} \cdot \frac{1}{2^{\frac{m}{n}}} = \int_0^{\frac{\pi}{2}} (\cos\varphi)^{\frac{m}{n}} \cos\frac{m}{n}\varphi . d\varphi$$

Soit $\frac{\varphi}{n} = \theta$, on aura

$$\frac{\pi}{2n} \cdot \frac{1}{2^{\frac{m}{n}}} = \int_0^{\frac{\pi}{2n}} (\cos n\theta)^{\frac{m}{n}} \cos m\theta . d\theta;$$

or $\cos n\theta = (\cos\theta)^n - \frac{n(n-1)}{2}(\cos\theta)^{n-2}.\sin^2\theta + \frac{n(n-1)(n-2)(n-3)}{2.3.4}(\cos\theta)^{n-4}.\sin^4\theta + \ldots$

donc en faisant $\cos\theta = y$, $d\theta = -\frac{dy}{\sqrt{(1-y^2)}}$,

$$\frac{\pi}{2n} \cdot \frac{1}{2^{\frac{m}{n}}} = -\int_1^{\cos\frac{\pi}{2n}} \sqrt[n]{(\psi y)^m} . fy . \frac{dy}{\sqrt{(1-y^2)}}$$

où $\psi y = y^n - \frac{n(n-1)}{2} y^{n-2}(1-y^2) + \frac{n(n-1)(n-2)(n-3)}{2.3.4} y^{n-4}(1-y^2)^2 - \ldots$

$fy = y^m - \frac{m(m-1)}{2} y^{m-2}(1-y^2) + \frac{m(m-1)(m-2)(m-3)}{2.3.4} y^{m-4}(1-y^2)^2 - \ldots$

Soit par exemple $m = 1$, $n = 4$, on aura

$$\frac{\pi}{8} \cdot \frac{1}{\sqrt[4]{(2)}} = -\int_1^{\cos\frac{\pi}{8}} \sqrt[4]{(1-8y^2+8y^4)} . \frac{ydy}{\sqrt{(1-y^2)}}$$

Si l'on fait $y^2 = 1 - z^2$, on trouvera

$$\frac{\pi}{8} \cdot \frac{1}{\sqrt[4]{(2)}} = -\int_0^{\sin\frac{\pi}{8}} dz \sqrt[4]{(1 - 8z^2 + 8z^4)}.$$