Arithmeticity of discrete subgroups and related topics

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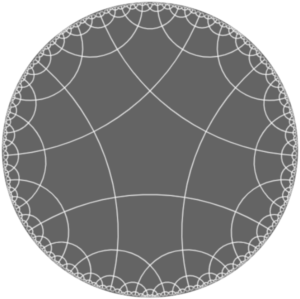
Discrete cocompact subgroups of $SL_2(\mathbb{R})$. Let G denote the quotient $PSL_2(\mathbb{R})$ of the group $SL_2(\mathbb{R})$ by the subgroup of scalar matrices. The group G is isomorphic to the group of orientation preserving isometries of the hyperbolic plane and also to the group of analytic automorphisms of the unit disc $\{z : |z| < 1\}$ in $\mathbb{C}$.



According to the classical uniformization theory, the space Ω_g of Riemann surfaces of genus $g > 1$ is naturally parametrized by conjugacy classes of discrete cocompact embeddings of Γ_g into G where Γ_g denotes the fundamental group of a Riemann surface of genus g. The space Ω_g is connected and has dimension $6g - 6$. Also every cocompact torsion free discrete subgroup of G is isomorphic to Γ_g for some $g > 1$.

It follows that "typically" a cocompact torsion free discrete subgroup of $SL_2(\mathbb{R})$ is not a conjugate of a subgroup consisting of matrices with algebraic coefficients.

Local rigidity. The situation dramatically changes if we replace the hyperbolic plane by higher dimensional symmetric spaces. As A. Selberg wrote in his 1960 paper "On discontinuous groups in higher dimensional symmetric spaces":

"For the higher-dimensional symmetric spaces, no uniformization theorem is available, and if we exclude certain product spaces, Euclidean spaces, or hyperbolic planes as factors, the only discontinuous groups known, that have compact fundamental domain, or at least finite volume of the fundamental domain are the groups that have arithmetic definition, or that are equivalent to these under inner automorphisms of the continuous group of isometries."

Selberg continues: *"One is therefore led to suspect that if one excludes certain product spaces, it might be true that there are no families of discontinuous groups with fundamental domain which is compact or of finite volume, and which depend continuously on a continuous parameter."*

The somewhat stronger phenomenon than the one described in the last quotation is now usually called *local rigidity*. Namely let Γ be a countable subgroup of a Lie group G. The subgroup Γ is called *locally rigid* if, for any continuous family $\{f_t, t \in [0,1]\}$ of homomorphisms of Γ into G such that $f_0(\gamma) = \gamma$ for $\gamma \in \Gamma$, one can find a continuous curve $\{g(t)\}$ in G such that

$$f_t(\gamma) = g(t)\gamma g(t)^{-1}$$

for all $\gamma \in \Gamma$ and all sufficiently small t.

Assume that $G = \mathbf{G}(\mathbb{R})$ where $\mathbf{G}$ is a linear algebraic group defined over $\mathbb{R}$. If Γ is finitely generated then the space $V_G(\Gamma)$ of homomorphisms of Γ into G is isomorphic to a real affine algebraic variety and therefore is locally path connected. Because of that , in the case of finitely generated subgroups Γ, the subgroup Γ is locally rigid if and only if for some neighborhood U in $V_G(\Gamma)$ of the identity embedding of Γ into G and any f in U there exists $g \in G$ such that $f(\gamma) = g\gamma g^{-1}$ for all $\gamma \in \Gamma$ (and moreover if U is "sufficiently small" then one can choose g to be "sufficiently close" to the identity element in G).

In the above mentioned paper Selberg proves local rigidity for discrete cocompact subgroups Γ of $G = SL_n(\mathbb{R})$ for $n \geq 3$, though he does not state it as a theorem.

Selberg uses local rigidity to prove that there exists $T \in G$ such that all elements of $T\Gamma T^{-1}$ are matrices with algebraic coefficients. Actually he considers the last statement as the main result of the paper.

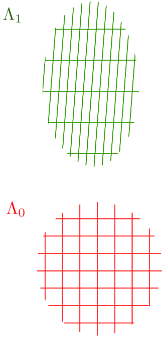
Also at the end of the paper Selberg conjectures that the eigenvalues of all elements in Γ are units in some algebraic number field and writes that "this problem seems also very difficult to prove".

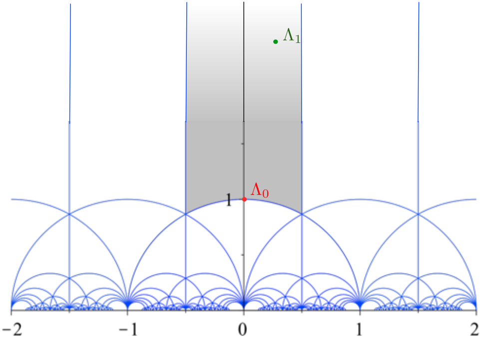
A very important feature of Selberg's proof of local rigidity is that he considers behavior under $\{f_t\}$ of elements of Γ which are "close to infinity". Mostow told me on several occasions that this feature in Selberg's proof was the initial inspiration for Mostow's asymptotic method to prove strong rigidity.

Shortly after Selberg proved local rigidity for discrete cocompact subgroups of $SL_n(\mathbb{R})$, A.Weil extended this result to other semisimple Lie groups using differential geometric methods. He proved that if G is a connected semisimple Lie group with trivial center and without compact factors, then a cocompact discrete subgroup Γ of G is locally rigid, provided $PSL_2(\mathbb{R})$ is not a direct factor of G which is closed modulo Γ.

Arithmetic lattices and the arithmeticity conjecture of Selberg and Piatetski-Shapiro. A discrete subgroup Γ of a Lie group G is called a *lattice* if the volume of G/Γ with respect to the Haar measure is finite. A lattice Γ is called *uniform* if G/Γ is compact and *non-uniform* otherwise.

The standard example of a lattice in $\mathbb{R}^n$ is $\mathbb{Z}^n$. All lattices in $\mathbb{R}^n$ are uniform and obtained by applying a linear transformation of $\mathbb{R}^n$ to $\mathbb{Z}^n$. A lattice Λ in $\mathbb{R}^n$ is called *unimodular* if the volume of $\mathbb{R}^n/\Lambda$ is equal to 1. The space Ω_n of unimodular lattices is naturally isomorphic to $SL_n(\mathbb{R})/SL_n(\mathbb{Z})$.





Lattices in $\mathbb{R}^2$ are parameterized by the modular surface $SL_2(\mathbb{R})/SL_2(\mathbb{Z})$. The grey region is a fundamental domain. Points high in the cusp, like Λ_1, correspond to tall thin lattices.

The space Ω_n is not compact. According to Mahler's compactness criterion, a subset A of Ω_n is relatively compact if and only if there exists $\varepsilon > 0$ such that $||v|| > \varepsilon$ for every non-zero vector v in every lattice $\Lambda \in A$.

On the other hand, the space $\Omega_n \cong SL_n(\mathbb{R})/SL_n(\mathbb{Z})$ has finite volume with respect to the Haar measure. Thus $SL_n(\mathbb{Z})$ is a non-uniform lattice in $SL_n(\mathbb{R})$.

Let **G** be a linear algebraic group defined over $\mathbb{R}$. The group $\mathbf{G}(\mathbb{R})$ of $\mathbb{R}$-points of **G** can be considered as an algebraic subgroup of $SL_m(\mathbb{R})$ for some m. If **G** is defined over $\mathbb{Q}$ (i.e. **G** is the set of zeroes of polynomials with rational coefficients) then the subgroup $\mathbf{G}(\mathbb{Z})$ consisting of all matrices in $\mathbf{G}(\mathbb{R})$ with integer coefficients is called an *arithmetic subgroup* of $\mathbf{G}(\mathbb{R})$.

Examples.

- $\mathbf{G} = SL_n$:
  $\mathbf{G}(\mathbb{R}) = SL_n(\mathbb{R})$, $n \times n$ real matrices of determinant 1.
  $\mathbf{G}(\mathbb{Z}) = SL_n(\mathbb{Z})$, integer matrices of determinant 1.
  $\mathbf{G}(\mathbb{R})/\mathbf{G}(\mathbb{Z}) = \Omega_n$, the space of unimodular lattices in $\mathbb{R}^n$.
- $\mathbf{G} = SO_Q$, where Q is a quadratic form of signature (p, q) in $\mathbb{R}^n$:
  $\mathbf{G}(\mathbb{R}) = SO_Q(\mathbb{R})$ are the linear transformations of $\mathbb{R}^n$ preserving the form Q.
  $\mathbf{G}(\mathbb{Z}) = SO_Q(\mathbb{Z})$ are the ones with integer coefficients (assuming Q has rational coefficients)
  $\mathbf{G}(\mathbb{R})/\mathbf{G}(\mathbb{Z})$ is the space of Q-orthogonal bases in $\mathbb{R}^n$ up to integral equivalence.
- $\mathbf{G} = Sp_{2n}$:
  $\mathbf{G}(\mathbb{R}) = Sp_{2n}(\mathbb{R})$, the $2n \times 2n$ real symplectic matrices.
  $\mathbf{G}(\mathbb{R})/\mathbf{G}(\mathbb{Z})$ is the space of symplectic bases in $\mathbb{R}^{2n}$ up to integral equivalence.

Any subgroup of $\mathbf{G}(\mathbb{R})$ commensurable with $\mathbf{G}(\mathbb{Z})$ is also called *arithmetic*. Recall that two subgroups are called *commensurable* if their intersection has finite index in each of them.

According to a theorem by A.Borel and Harish-Chandra, if $\mathbf{G}$ is semisimple, then any arithmetic subgroup Λ of $\mathbf{G}(\mathbb{R})$ is a lattice. The lattice Λ is uniform if and only if it does not contain non-trivial unipotent elements. Recall that a matrix is called *unipotent* if 1 is its only eigenvalue.

Let G denote the connected component of the identity of the group $\mathbf{G}(\mathbb{R})$. If H is another connected Lie group and $\varphi : G \to H$ is a continuous epimorphism with compact kernel, then $\varphi(\Lambda)$ is a lattice in H for any lattice Λ in G. The lattice $\varphi(\Lambda)$ is uniform if and only if Λ is uniform. If Γ is an arithmetic subgroup of G which is also a lattice then $\varphi(\Lambda)$ is called an *arithmetic lattice* in H.

Assume that the connected Lie group H is semisimple and has no compact factors. A lattice Δ in H is called *irreducible* if $\Delta \cdot F$ is not discrete in H for any infinite normal subgroup F of H. In the case when the center of H is trivial this condition is equivalent to another condition: $(H_1 \cap \Delta) \cdot (H_2 \cap \Delta)$ has infinite index in Δ for any non-trivial direct product decomposition $H = H_1 \times H_2$.

An arithmetic non-uniform lattice Δ in H contains non-trivial unipotent elements. If the center of H is trivial and the arithmetic lattice Δ is non-uniform and irreducible, then we can assume that the homorphism φ is an isomorphism for a suitable choice of $\mathbf{G}$.

Also if the center of H is trivial and a lattice Γ in H is non-uniform and irreducible, there is the following criterion for the lattice Γ to be arithmetic. Let $\Gamma^{(u)}$ denote the set of unipotent elements in Γ and let $\log : \Gamma^{(u)} \to \mathfrak{H}$ be the natural logarithmic map from $\Gamma^{(u)}$ to the Lie algebra $\mathfrak{H}$ of H. Then Γ is arithmetic if and only if the $\mathbb{Z}$-span of $\log(\Gamma^{(u)})$ is a lattice in $\mathfrak{H}$.

Let G be a connected semisimple Lie group without compact factors and with trivial center. For quite a long time there were examples of non-arithmetic irreducible lattices in G only in the case when G is locally isomorphic to $SL_2(\mathbb{R})$. This led Selberg to conjecture that, with few exceptions, all irreducible lattices in G are arithmetic. Initially the only exceptions were groups locally isomorphic to $SL_2(\mathbb{R})$. In view of the results of Selberg and Weil mentioned above, if G is not locally isomorphic to $SL_2(\mathbb{R})$ then any uniform irreducible lattice Γ in G is of the form $\Gamma = g\Lambda g^{-1}$ where $g \in G$ and Λ consists of matrices with algebraic coefficients. At first glance, this result seems to imply that Γ is arithmetic. But this impression is misleading.

Around 1965 Makarov and Vinberg gave examples of non-arithmetic lattices in G when $G = PSO(n,1)^0$ is the group of orientation preserving isometries of the n-dimensional hyperbolic space for n equal 3, 4, or 5. The examples by Makarov and Vinberg are groups generated by reflections. Vinberg also gave a criterion of the arithmeticity of a lattice in $G = PSO(n,1)^0$ generated by reflections.

The $\mathbb{R}$-*rank* of a semisimple Lie group G without compact factors is defined as the dimension of a maximal diagonalizable over $\mathbb{R}$ subgroup of G or, equivalently, as the dimension of a maximal totally geodesic Euclidean subspace of the symmetric space corresponding to G. The $\mathbb{R}$-rank of $SL_n(\mathbb{R})$ is $n-1$, and the $\mathbb{R}$-rank of $SO(n,1)$ is 1. The groups with $\mathbb{R}$-rank greater than 1 are called groups of *higher rank*.

Eventually Selberg conjectured that if G is of higher rank then all irreducible non-uniform lattices in G are arithmetic. For uniform lattices Γ, Selberg conjectured only that the eigenvalues of elements of Γ are algebraic integers, and in this case the conjecture is due to Piatetski-Shapiro. The difference between non-uniform and uniform lattices can be explained by the fact that if Γ is an arithmetic non-uniform irreducible lattice in G then, in some basis of the Lie algebra of G, the group $\mathrm{Ad}(\Gamma)$ consists of matrices with coefficients in $\mathbb{Z}$.

Arithmeticity of non-uniform lattices. Selberg and Piatetski-Shapiro developed a strategy to prove the Selberg conjecture about the arithmeticity of a non-uniform lattice Γ in a semisimple Lie group G of $\mathbb{R}$-rank greater than 1. This strategy is based on the study of unipotent elements in Γ and subgroups of Γ asociated with them.

In the paper "Recent developments in the theory of discontinuous groups of motions of symmetric spaces" written in 1968 and published in 1970, Selberg considered (among other topics) the case when G is the product $SL_2(\mathbb{R})^r$ of $r \geq 2$ copies of $SL_2(\mathbb{R})$. The group $SL_2(K)$ is naturally embedded into G and, under this embedding, $SL_2(L)$ is an arithmetic lattice in G, where K is a totally real extension of $\mathbb{Q}$ of degree r and L denotes the ring of algebraic integers in K.

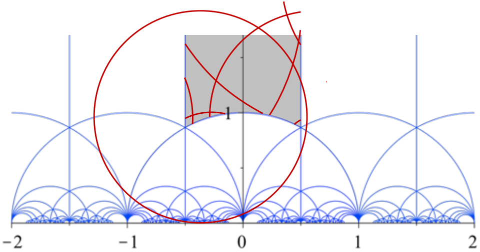
Let T denote the group of upper triangular matrices in G. In 1959 Piatetski-Shapiro showed that if Γ contains a non-trivial unipotent element then, for a suitable choice of K and after the conjugation of Γ by an element g of G, the subgroups $T \cap \Gamma$ and $T \cap SL_2(L)$ are commensurable. In 1964 Selberg proved that Γ indeed contains a non-trivial unipotent element. Combining these two results and using the condition $r \geq 2$, Selberg shows that, for any $\gamma_1, \gamma_2 \in \Gamma$, the intersections of the subgroup $\gamma_1 T \gamma_1^{-1} \cap \gamma_2 T \gamma_2^{-1}$ with Γ and with $SL_2(L)$ are commensurable. This easily implies that $\Gamma \cap SL_2(K)$ has finite index in Γ ("rationality" of Γ). After that the proof of the arithmeticity of Γcan be completed by using results about subgroups generated by unipotent elements in arithmetic groups.

For general G the strategy is similar but the proof is much more complicated. The first step was done in 1968 in the paper "Proof of Selberg's conjecture" by Kazhdan and myself, where we proved the existence of non-trivial unipotent elements Γ for any non-uniform lattice Γ in G. The subgroup T of triangular matrices in $SL_2(\mathbb{R})^r$ and its conjugates are replaced by parabolic subgroups of G. It was first realized by Piatetski-Shapiro that the following statement should play an important role in some approaches to the proof of arithmeticity of non-uniform lattices:

(D) *Let* $\{u(t)\}$ *be a one-parameter group of unipotent linear transformations of* $\mathbb{R}^n$ *and let* $\Lambda \in \Omega_n$. *Then, there exists* $\varepsilon > 0$ *such that the set*

$$\{t \in \mathbb{R} : ||u(t)v|| > \varepsilon \text{ for every non-zero } v \in \Lambda\}$$

is not bounded.



Statement (D) illustrated for $n = 2$: The red horocycle is a unipotent orbit whose infinitely many recurrences to a compact set in Ω_2 are indicated in a single fundamental domain.

The statement (D) was conjectured (or, more precisely, stated as a theorm without a proof) by Piatetski-Shapiro and slightly later conjectured independently by Garland and Raghunathan. I proved (D) in a paper published in 1969. Using (D), I was able to give the proof of the "rationality" part in a long paper submitted for publication in 1971 and published only in 1975. A couple of years later I completed the proof of arithmeticity of non-uniform lattices using results about sugroups generated by unipotent elements in arithmetic groups. M.S.Raghunathan couldn't prove the statement (D) but he was able to prove "rationality" without using (D) (in a long paper submitted for publication in 1973 and published in 1975).

The statement (D) looks quite technical. But it and especially the method of its proof became quite influential.
M.S.Raghunathan wrote me that the analysis of my proof of (D) was one of the inspirations for stating his conjecture about the closures of orbits of unipotent flows on homogeneous spaces. Raghunathan's conjecture was proved in a very special case in my work on the Oppenheim conjecture and in some other special cases in a series of joint papers by S.G.Dani and myself. In general case Raghunathan's conjecture and its quantitative analogs were proved by M.Ratner in a fundamental series of papers. Also it should be noticed that the quantitative generalization of the statement (D) played a significant role in various approaches to the proof of the Raghunathan conjecture.

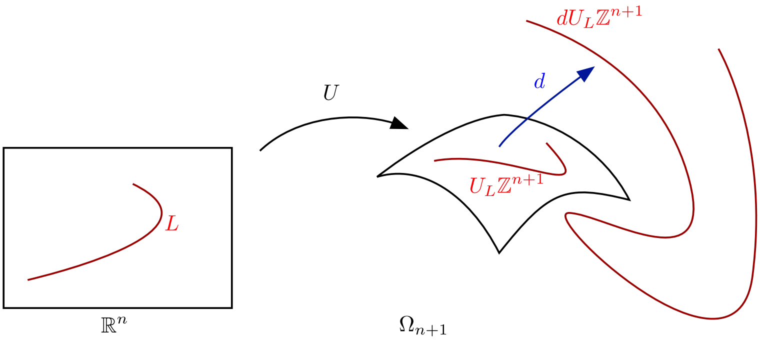
S.G.Dani significantly modified and simplified the proof of the statement (D). It also turned out that the method of the proof of (D) can be applied to obtain results on "the behavior at infinity" of various classes of curves in Ω_n. As a consequence of these results for some special class of curves D.Kleinbock and I proved in a 1998 paper some long standing Sprindzuk's conjectures in the theory of Diophantine approximation on manifolds.

Roughly speaking, these conjectures say that almost all points on any analytic curve L in $\mathbb{R}^n$ have same Diophantine properties as almost all points in $\mathbb{R}^n$ if L is not contained in any codimension 1 hyperplane.

For a vector $\mathbf{v} = (v_1, ..., v_n) \in \mathbb{R}^n$, let $U_{\mathbf{v}} \in (SL_{n+1}(\mathbb{R})$ denote the unipotent matrix

$$\begin{bmatrix} 1 & v_1 & v_2 & \cdots & v_n \\ 0 & 1 & 0 & \cdots & 0 \\ \vdots & & & & \vdots \\ 0 & 0 & 0 & \cdots & 1 \end{bmatrix}$$

S.G.Dani established a correspondence between Diophantine properties of $\mathbf{v}$ and the rate of "divergence to infinity" in Ω_{n+1} of orbits of $U_{\mathbf{v}}\mathbb{Z}^{n+1}$ under certain diagonal subsemigroups of $SL_{n+1}(\mathbb{R})$.



Let $U_L \subset SL_{n+1}(\mathbb{R})$ be the image of the curve L under the map $\mathbf{v} \to U_{\mathbf{v}}$. We consider the curve $U_L\mathbb{Z}^{n+1}$ in Ω_{n+1} and the translations of this curve by diagonal $d \in SL_{n+1}(\mathbb{R})$.

For certain d we provide "good upper estimates" for the measure of the intersection of these translations with "neighborhoods of infinity". (To get these estimates we use modifications of the arguments used in the proof of quantitative versions of (D).) Combining these estimates with Dani correspondence, we easily get the desired result.

Strong rigidity. Selberg and Weil proved local rigidity which says that deformations of uniform lattices can be extended to continuous homomorphisms of ambient semisimple Lie groups. In late 60s and early 70s Mostow proved that the same is true for all isomorphisms between uniform lattices: this was quite unexpected. Mostow called this phenomenon *strong rigidity*. There are two equivalent formulations of Mostow's strong ridity theorem, one geometric and another in the language of Lie groups.

Let Y be a locally symmetric Riemannian space. Assume that the sectional curvature of Y at every point is non-positive. Moreover assume that Y is compact and connected. Then the fundamental group $\pi_1(Y)$ determines Y uniquely up to an isometry and choice of normalized constants, provided that Y has no closed one or two dimensional geodesic subspaces which are direct factors locally.

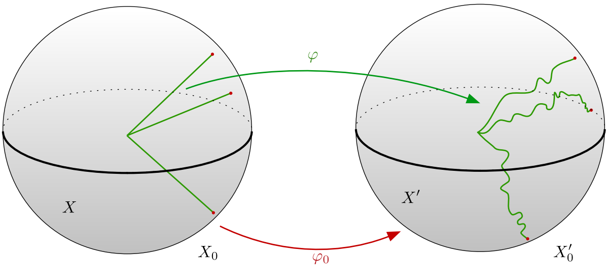
In particular, if $n \geq 3$ then two compact n-dimensional Riemannian spaces of constant curvature -1 with isomorphic fundamental groups are isometric.

Let G be a connected semisimple Lie group with trivial center and with no non-trivial compact normal subgroups. Let Γ be a uniform lattice in G. Then the pair (G, Γ) uniquely determined by Γ, provided $PSL_2(\mathbb{R})$ is not a direct factor of G which is closed modulo Γ.

That is, given two such pairs (G, Γ) and (G', Γ'), and an isomorphism $\theta : \Gamma \to \Gamma'$ there exists a continuous isomorphism $\bar{\theta} : G \to G'$ such that θ is the restriction of $\bar{\theta}$ to Γ provided there is no direct factor H of G isomorphic to $PSL_2(\mathbb{R})$ such that ΓH is a closed subgroup of G.

Let X (resp. X') be the symmetric space corresponding to G (resp. G'). The group G (resp. G') transitively acts on X (resp. X') by isometries. It is easy to show that one can find a continuous map $\varphi : X \to X'$ which is Γ-*equivariant*, i.e. $\varphi(\gamma x) = T(\gamma)\varphi(x)$ for all $\gamma \in \Gamma$ and $x \in X$, Since G/Γ is compact, the map φ is uniformly continuous. It is not necessarily a homeomorphism, but it is a *pseudo-isometry* which, roughly speaking, means that it is bi-Lipschitz on "large scale."

The main idea of Mostow is to study the asymptotic behavior of the map φ. He first shows that φ extends by continuity to a Γ-equivariant homeomorphism $\varphi_0 : X_0 \to X_0'$ between the Furstenberg-Satake boundaries X_0 of X and X_0' of X'. After that Mostow essentially proves that φ_0 is a rational (algebraic) map of algebraic varieties. The rest of the proof is quite easy.



The Furstenberg-Satake boundary X_0 of X can be described as the quotient G/P of G by a minimal parabolic subgroup P. If X is the n-dimensional hyperbolic space then X_0 is the $(n-1)$-dimensional sphere. If $G = SL_n(\mathbb{R})$ then X_0 is the flag variety or, equivalently, the quotient of G by the subgroup of all upper triangular matrices in G. If $\mathbb{R}$-rank of G is 1, then $X \cup X_0$ is a compactification of X. For groups G of higher rank, X_0 is the orbit of the smallest dimension for the action of the group G on the Furstenberg-Satake compactification $\bar{X}$ of X. The space $\bar{X}$ is the disjoint union of finitely many orbits of which only X_0 is compact.

The existence of φ_0 is based on geometric and topological arguments, most notably on generalizations of Morse lemma about asymptotic behavior of quasi-geodesics.

Mostow proves that φ_0 is an isomorphism of Tits geometries. For groups G of higher rank this proves the rationality of φ_0 due to the Tits theorem about isomorphisms of Tits geometries in the higher rank case (this Tits theorem is a generalization of the Fundamental Theorem of Projective Geometry).

If X is the n-dimensional hyperbolic space, $n \geq 3$, the map φ_0 is quasiconformal and therefore almost everywhere differentiable with non-degenerate differential. Using this and an ergodicity argument, Mostow completes the proof of rationality in the case of hyperbolic spaces. For other groups G of $\mathbb{R}$-rank 1 the proof of rationality is similar.

Arithmeticity of uniform lattices and superrigidity. Let G be a connected semisimple Lie group with trivial center and without compact factors. Let Γ be an irreducible uniform lattice in G. Let us denote the $\mathbb{R}$-rank of G by $\text{rank}_{\mathbb{R}} G$. In 1973 I proved the arithmeticity of Γ for higher rank groups G using the following *superrigidity theorem* :

Let k be a locally compact field of characteristic zero, and let $T : \Gamma \to SL_n(k)$ be a homomorphism. Assume that the Zariski closure $\mathbf{H}$ *of $T(\Gamma)$ is connected, absolutely simple, and has trivial center. Also assume that* $\text{rank}_{\mathbb{R}} G > 1$.

(i) *If k is a finite extension of $\mathbb{Q}_p$ then $T(\Gamma)$ is bounded in k-topology.*

(ii) *If k is $\mathbb{R}$ or $\mathbb{C}$ and $T(\Gamma)$ is not bounded in k-topology then T extends to a continuous homomorphism of G to $SL_n(k)$.*

The statement (i) (resp. (ii)) is called *non-Archimedean superrigidity* (resp. *Archimedean superrigidity*). We can assume that $\Gamma \subset SL_n(K)$ where K is a finite extension of $\mathbb{Q}$. Every embedding σ of K into a locally compact field k induces a homomorphism $T_\sigma : \Gamma \to SL_n(k)$.

Roughly speaking, the arithmeticity of Γ is proved by applying the superrigidity theorem to such homomorphisms. To prove that eigenvalues of all elements of γ are units of some algebraic number field, it is enough to use only non-Archimedean superrigidity.

In the same way as strong rigidity can be considered to be a generalization of local rigidity, superrigidity can be considered to be a generalization of strong rigidity. The proof of the superrigidity theorem is based on two statements (A) and (B) about existence and rationality of Γ-equivariant mesurable maps.

An algebraic action of an algebraic group $\mathbf{F}$ on an algebraic variety $\mathbf{M}$ is called *strongly effective* if $\mathbf{F}$ acts effectively on every orbit $\mathbf{F}x, x \in \mathbf{M}$. Since $\mathbf{H}$ is absolutely simple, this condition for actions of the group $\mathbf{H}$ is equivalent to the condition that the stabilizer in $\mathbf{H}$ of every point $x \in \mathbf{M}$ is trivial.

Let P denote a minimal parabolic subgroup of G. The quotient G/P is the Furstenberg-Satake boundary. .

(A) *If* $T(\Gamma)$ *is not bounded in* k*-topology then one can find a* k*-rational strongly effective action of* $\mathbf{H}$ *on an algebraic* k*-variety* $\mathbf{M}$ *and a measurable map* $\varphi : G \to \mathbf{M}(k)$ *such that*

$$\varphi(\gamma gp) = T(\gamma)\varphi(g)$$

for all $\gamma \in \Gamma, p \in P$ *and almost all* $g \in G$.

The map φ induces a measurable map $\varphi' : G/P \to \mathbf{M}(k)$ which is Γ-equivariant in the sense that $\varphi'(\gamma x) = T(\gamma)\varphi'(x)$ for all $\gamma \in \Gamma$ and all $x \in G/P$. The first proof of (A) used Oseledec multiplicative ergodic theorem and some results from representation theory. Furstenberg gave an alternative proof based on his boundary theory.

(B) *Suppose that* $\mathbf{H}$ *acts* k*-rationally on an algebraic* k*-variety* $\mathbf{M}$. Let $\varphi : G \to \mathbf{M}(k)$ *be a measurable map such that*

$$\varphi(\gamma g p) = T(\gamma)\varphi(g)$$

all $\gamma \in \Gamma$ *almost all* $g \in G$. *Assume that* $\operatorname{rank}_{\mathbb{R}} G > 1$. *Then* φ *is a rational (algebraic) map*.

The proof of (B) is based on the combination of methods from ergodic theory/measure theory and the algebraic group theory. One of the ingredients of the proof is a generalization of a classical fact that a function $f(x, y)$, which is rational in each variable when another variable is fixed, is rational.

Selberg's proof of local rigidity. Mostly algebraic methods. Implicitly uses Poincare recurrence theorem.

Mostow's proof of strong rigidity. Geometric and topological methods. Isomorphisms of Tits geometries. Quasiconformal maps and its generalizations. Uses an ergodicity argument.

Superrigidity. Combination of methods from ergodic theory/probability and the theory of algebraic groups.

For groups G of $\mathbb{R}$-rank 1, the lattice Γ is not necessarily arithmetic. There are three infinite series of groups G of $\mathbb{R}$-rank 1 and also one exceptional group. They correspond respectively to

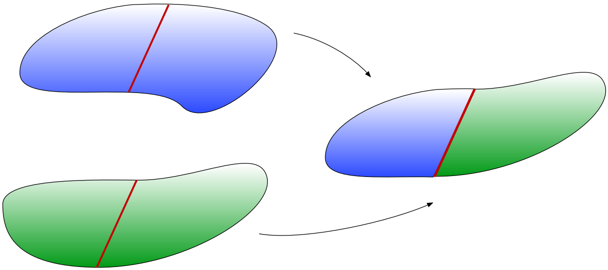
(i) real hyperbolic spaces of dimension n,

(ii) complex hyperbolic spaces of complex dimension n,

(iii) quaternionic hyperbolic space of quaternionic dimension n,

(iv) the octonian hyperbolic space.

In the case (i) there are earlier mentioned examples by Makarov and Vinberg of non-arithmetic lattices in dimensions 3, 4 and 5. In 1988 Gromov and Piatetski-Shapiro gave examples of non-arithmetic lattices for all n using so called *hybrid construction*.



In the case (ii), there are examples of non-arithmetic lattices in complex dimensions 2 and 3. In 1980 Mostow gave examples of non-arithmetic lattices for $n = 2$ using groups generated by complex reflections. Using monodromies of hypergeometric functions, Deligne and Mostow constructed in 1986 examples of non-arithmetic lattices for $n = 2$ and $n = 3$.

For $n \geq 4$ it is not known if non-arithmetic lattices exist. On the other hand using earlier resilts by H.Esnault and N.Groechenig, G.Baldi and E.Ullmo proved in 2020 that for every $n > 1$ the eigenvalues of elements of Γ are algebraic integers.

In the cases (iii) and (iv), around 1990 K.Corlette proved Archimedean superrigidity using harmonic maps. Shortly after that, Gromov and Schoen proved non-Archimedean supperigidity using a similar approach. Thus in cases (iii) and (iv) all lattices Γ are arithmetic.

Let

$$\mathrm{Comm}_G(\Gamma) = \{g \in G : g\Gamma g^{-1} \text{and}\, \Gamma \text{are commensurable}\}$$

denote the *commensurator* of Γ in G. If $\mathbf{H}$ is a connected algebraic $\mathbb{Q}$-group then $\mathbf{H}(\mathbb{Q})$ is dense in $\mathbf{H}(\mathbb{R})$. It easily implies that if the lattice Γ is arithmetic then $\mathrm{Comm}_G(\Gamma)$ is dense in G. The converse is also true. I proved this at the same time as I proved arithmeticity of Γ for the case $\mathrm{rank}_{\mathbb{R}} > 1$. The arithmeticity again is deduced from some version of superrigidity, and the proof of superrigidity follows the same strategy as in the case $\mathrm{rank}_{\mathbb{R}} G > 1$. Using harmonic maps, A.Karlsson, T.Gelander and I gave in 2006 a relatively short proof of superrigidity in the case when $\mathrm{Comm}_G(\Gamma)$ is dense in G.

At the beginning of 2019 answering a question by A. Reid and C. McMullen, A.Mohammadi and I proved the following theorem:

Let $M = \mathbb{H}^3/\Gamma$ be a closed hyperbolic 3-manifold. If M contains infinitely many totally geodesic surfaces, then M is arithmetic. That is Γ is arithmetic.

Our proof uses, among other things, the multiplicative ergodic theorem and the martingale convergence theorem. Shortly after we proved the above theorem, Bader, Fisher, Miller and Stover proved that if a finite volume hyperbolic manifold $\mathbb{H}^n/\Gamma$ contains infinitely many maximal totally geodesic subspaces of dimension at least two, then Γ is arithmetic. Their proof and ours both use a superrigidity theorem to prove arithmeticity, but their proof and ours of the superrigidity theorem are quite different.