



THE
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A glimpse of the Laureates' work

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The 2015 Abel Prize celebrates two mathematicians, John F. Nash Jr. and Louis Nirenberg, and the interconnections of their work. Although the two men never formally collaborated, they were influences on each other during the mid-1950s when both lived in New York. This year's prize also tells the wider story about the interconnections between two fields of mathematics.

The first field is geometry. The ancient Greek historian Herodotus was the first person to write the word 'geometry', meaning earth-measure, using it to describe how the Egyptians calculated areas of fertile land. The Greeks turned geometry into a rigorous discipline about the behaviour of points, lines and planes. In Euclid's book *Elements*, for example, we learn that the angles in a triangle always add up to 180 degrees.

In the beginning of the nineteenth century, however, mathematicians began to think about the shape of space in a way that was heretical to the Greeks. In some cases, Euclid's rules did not apply. When you draw a triangle on a ball, for example, the angles will always add up to more than 180 degrees, and when you draw a triangle on the inside of a saddle, the angles add up to less than 180 degrees. Which rules were the right ones?

To answer this problem, mathematicians developed the concept of curvature. When the curvature of a surface is zero, it is flat, or Euclidean,

and Euclid's assumptions hold. When curvature is positive, however, the surface folds in on itself, like the surface of a ball, and when curvature is negative, the surface billows out, like the surface of a saddle. In both these cases the rules of Euclid are false.

The German mathematician Carl Friedrich Gauss in 1827 was the first person to define terms and prove theorems about the curvature of surfaces. Less than two decades later his student Bernhard Riemann made a significant advance by introducing a new, highly abstract way to think about surfaces that disconnected them from their existence in any exterior world. For Riemann, the curvature at each point on a surface was defined purely by the rules for measuring distance and calculating angles around that point. He also generalized his theory from two dimensions to an unlimited number of dimensions. Riemann's terminology and framework were revolutionary. They marked the birth of modern geometry, and the shockwaves were felt in many other areas. For example, many years later Einstein used Riemann's ideas in his theory of general relativity, in which the geometry of four dimensional space-time is curved.

The second field that relates to the Laureates' work is partial differential equations. Its history dates back to Isaac Newton and Gottfried Leibniz who in the seventeenth century independently devised a method – "calculus" – to calculate the



instantaneous rates of change of varying quantities. Newton required calculus to describe the speed and acceleration of physical objects and thus develop his laws of motion and gravity.

The instantaneous rate of change of a variable is known as its 'differential', and so equations that involve rates of change are called differential equations. When a mathematical quantity depends on two variables, such as temperature depending on time and position, the "partial differentials" correspond to the rates of change with respect to one variable or the other. Equations with partial differentials are called "partial differential equations", or PDEs, and they dominate applied mathematics. Maxwell's equations about electromagnetism, the Navier-Stokes equations about fluid dynamics and the Schrödinger equation about quantum mechanics are all PDEs.

As well as having different histories and purposes, geometry and PDEs are stylistically very different types of mathematics. Geometry is often visual, intuitive and big picture, whereas the study of PDEs require a passionate attention to the tiniest details. The work of the Laureates shows, however, that the fields can join up in fruitful ways.

As stated above, Riemann's surfaces don't exist in the real world, by which we mean the three-dimensional Euclidean world. They are abstractly-defined geometrical objects that may be impossible to visualise in three Euclidean dimensions. An obvious question to mathematicians was whether or not it was possible to "embed" these surfaces in Euclidean space. In an embedding, every point on the abstract surface is mapped to a point on a surface in the real world. When the embedding is 'isometric',

distances are maintained, meaning that the distance between any two points on the abstract surface is equal to the distance between those two points on the mapping. Isometric embedding, in lay terms, allows us to take a slippery, abstract geometrical idea and to make it concrete.

Nirenberg proved that it is possible to embed a sphere – that it is, a sphere defined as a 2-dimensional Riemannian surface with positive curvature – isometrically as a convex surface in three Euclidean dimensions. Nash's isometric embedding theorems are more general. He proved that any Riemannian surface can be made concrete in a Euclidean world, although sometimes one requires more than three dimensions to do it.

When you prove an embedding theorem, you must equate how you move around the abstract surface with how you move around the Euclidean surface, and this gives rise to PDEs. The technique invented by Nash to solve the PDE in his embedding theorem was so innovative and useful that his results have had more repercussions within the field of PDEs than within geometry – even though the problem was originally a geometrical one

The embedding theorems of Nirenberg and Nash are only a small part of their collective work, and only part of what the Abel prize is rewarding, yet these results have provided a foundation for others who have continued to discover the richness of using PDEs within geometry, especially in recent years. One example is Grigori Perelman, who combined these two fields in his proof of the Poincaré conjecture, which was one of the most famous open problems in mathematics.

