THE ABEL PRIZE 2018

A glimpse of the Laureate's work

By Alex Bellos

The citation for Robert P. Langlands' Abel Prize begins:

The Langlands program predicts the existence of a tight web of connections between automorphic forms and Galois groups.

In order to understand the significance of the Langlands program we need to consider the mathematical histories of both of these concepts: automorphic forms and Galois groups.

Harmonic analysis

Automorphic forms come from an area of mathematics called harmonic analysis, which can be thought of, in very approximate and general terms, as the study of periodic waves. The best-known periodic wave is the sine wave, which consists of a peak and a trough repeating ad infinitum. The use of the word harmonic here comes from the sine wave's application to the physics of sound. A musical note, for example, a C played by a violin, can be understood as the superposition of many sine waves, each of these waves representing a "harmonic" of the C. Automorphic forms are a generalization of the idea of periodic waves, expressed using a more sophisticated geometrical language.

Number theory

Galois groups are a concept that emerged from number theory, the study of the properties of numbers. One important topic within number theory is how to solve polynomial equations, meaning equations where the exponents are positive whole numbers. For example, how do we solve the following equation?

$x^2 + x + 1 = 0$

This polynomial is a quadratic equation and its two solutions be calculated using the quadratic formula known to most school children. The solutions are $x = -½ + (\sqrt{-3})/2$ and $x = -½ - (\sqrt{-3})/2$.

The solutions look similar: the first part in both cases is ½, and the second is either plus or minus $(\sqrt{-3})/2$. In other words, they display a symmetry: you just flip the sign from plus to minus or from minus to plus to get from one to the other. In the early nineteenth century, the French mathematician Evariste Galois studied the symmetries between solutions of polynomial equations. The table of these symmetrical relations is now called a Galois group.

The connections

The Langlands program is a hugely ambitious project to bridge harmonic analysis and number theory, initially by showing deep connections between automorphic forms and Galois groups. Harmonic analysis and number theory are two separate fields, each with their own concepts, techniques and terminology; the program reveals powerful equivalences between them.

One basic concept crucial to Langlands' ideas is modular arithmetic, a way of doing arithmetic with a fixed set of consecutive numbers such that when you count beyond the top you start from zero again. One example of modular arithmetic is the 12–hour clock. If it is 11 o'clock and you add 5 hours, using normal arithmetic you would get to 16 o'clock. But there is no 16 in the 12–hour clock! We all know that 11 o'clock plus 5 hours is 4 o'clock, since once you hit 12 you start from zero again.

In his *Disquisitiones Arithmeticae* (1801), the German mathematician Carl Friedrich Gauss established a theory of modular arithmetic and presented as its "fundamental theorem" the law of quadratic reciprocity, which is about the solvability of quadratic equations using modular arithmetic. Let's look again at the quadratic equation above $x^2 + x + 1 = 0$. If we are considering modular arithmetic with a modulus of 3, meaning we are using only the three numbers 0, 1 and 2, then this equation has a solution of $x = 1$, since $1^2 + 1 + 1 = 3$, which is the same as 0 when the modulus is 3. Since Gauss, mathematicians have been interested in how the solvability of certain types of equation depends on the modulus and how this relates to the Galois groups of these equations.

A specific case of how the Langlands program connects number theory and harmonic analysis can be seen by considering a type of polynomial equation called an "elliptic curve". If you take an elliptic curve and find the number of solutions it has for every modulus when the modulus is a prime number (that is, the numbers 2, 3, 5, 7, 11, … , which are those numbers only divisible by themselves and 1) you will generate a sequence of numbers. This sequence of numbers, however, can also be generated by a different type of mathematical object that is (very approximately) analogous to a periodic wave, and can be investigated using the tools of harmonic analysis.

In his 1967 letter to André Weil and in his 1970 *Problems in the Theory of Automorphic Forms*, Langlands made many wide-reaching conjectures that link number theory and harmonic analysis, which most experts believe are true but many of which have not yet been proved. Even so, it has been one of the most fertile areas for mathematical research. In 2002 and 2010 mathematicians were awarded the Fields Medal for proving Langlands' conjectures.

The Langlands program is exciting for mathematicians because it bridges apparently unrelated disciplines, revealing a deeper structure underlying all mathematics and providing new ways to solve intractable problems. But it is also intoxicating because of the nature of the connections: number theory is an area where numbers often appear with no predictable order, yet automorphic forms are full of smooth curves, regular patterns and beautiful symmetries.