The Abel Prize Laureate 2019

Karen Keskulla Uhlenbeck
University of Texas at Austin

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Karen Keskulla Uhlenbeck receives the 2019 Abel Prize

for her pioneering achievements in geometric partial differential equations, gauge theory and integrable systems, and for the fundamental impact of her work on analysis, geometry and mathematical physics.
The Norwegian Academy of Science and Letters has decided to award the Abel Prize for 2019 to

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Karen Keskulla Uhlenbeck is a founder of modern Geometric Analysis. Her perspective has permeated the field and led to some of the most dramatic advances in mathematics in the last 40 years.

Geometric analysis is a field of mathematics where techniques of analysis and differential equations are interwoven with the study of geometrical and topological problems. Specifically, one studies objects such as curves, surfaces, connections and fields which are critical points of functionals representing geometric quantities such as energy and volume. For example, minimal surfaces are critical points of the area and harmonic maps are critical points of the Dirichlet energy. Uhlenbeck’s major contributions include foundational results on minimal surfaces and harmonic maps, Yang-Mills theory, and integrable systems.

**Minimal surfaces and bubbling analysis**

An important tool in global analysis, preceding the work of Uhlenbeck, is the Palais—Smale compactness condition. This condition, inspired by earlier work of Morse, guarantees existence of minimisers of geometric functionals and is successful in the case of 1-dimensional domains, such as closed geodesics.

Uhlenbeck realised that the condition of Palais—Smale fails in the case of surfaces due to topological reasons. The papers of Uhlenbeck, co-authored with Sacks, on the energy functional for maps of surfaces into a Riemannian manifold, have been extremely influential and describe in detail what happens when

© In 1987 Karen K. Uhlenbeck moved to the University of Texas at Austin to take up the Sid W. Richardson Foundation Regents’ Chair in mathematics. She would remain at the University of Texas until the end of her working career. Currently, Uhlenbeck is a Visiting Senior Research Scholar at Princeton University as well as a Visiting Associate at the Institute for Advanced Study (IAS).

PHOTO: Marsha Miller
the Palais-Smale condition is violated. A minimising sequence of mappings converges outside a finite set of singular points and by using rescaling arguments, they describe the behaviour near the singularities as bubbles or instantons, which are the standard solutions of the minimising map from the 2-sphere to the target manifold.

In higher dimensions, Uhlenbeck in collaboration with Schoen wrote two foundational papers on minimising harmonic maps. They gave a profound understanding of singularities of solutions of non-linear elliptic partial differential equations. The singular set, which in the case of surfaces consists only of isolated points, is in higher dimensions replaced by a set of co-dimension 3.

The methods used in these revolutionary papers are now in the standard toolbox of every geometer and analyst. They have been applied with great success in many other partial differential equations and geometric contexts. In particular, the bubbling phenomenon appears in many works in partial differential equations, in the study of the Yamabe problem, in Gromov’s work on pseudoholomorphic curves, and also in physical applications of instantons, especially in string theory.

**Gauge theory and Yang-Mills equations**

After hearing a talk by Atiyah in Chicago, Uhlenbeck became interested in gauge theory. She pioneered the study of Yang-Mills equations from a rigorous analytical point of view. Her work formed a base of all subsequent research in the area of gauge theory.

Gauge theory involves an auxiliary vector bundle over a Riemannian manifold.

The basic objects of study are connections on this vector bundle. After a choice of a trivialisation (gauge), a connection can be described by a matrix valued 1-form. Yang-Mills connections are critical points of gauge-invariant functionals. Uhlenbeck addressed and solved the fundamental question of expressing Yang-Mills equations as an elliptic system, using the so-called Coulomb gauge. This was the starting point for both Uhlenbeck’s celebrated compactness theorem for connections with curvature bounded in $L^p$, and for her later results on removable singularities for Yang-Mills equations defined on punctured 4-dimensional balls. The removable singularity theory for Yang-Mills equations in higher dimensions was carried out much later by Gang Tian and Terence Tao. Uhlenbeck’s compactness theorem was crucial in Non-Abelian Hodge Theory and, in particular, in the proof of the properness of Hitchin’s map and Corlette’s important result on the existence of equivariant harmonic mappings.

Another major result of Uhlenbeck is her joint work with Yau on the existence of Hermitian Yang-Mills connections on stable holomorphic vector bundles over complex n-manifolds, generalising an earlier result of Donaldson on complex surfaces. This result of Donaldson-Uhlenbeck-Yau links developments
in differential geometry and algebraic geometry, and is a foundational result for applications of heterotic strings to particle physics.

Uhlenbeck’s ideas laid the analytic foundations for the application of gauge theory to geometry and topology, to the important work of Taubes on the gluing of self-dual 4-manifolds, to the groundbreaking work of Donaldson on gauge theory and 4-dimensional topology, and many other works in this area. The book written by Uhlenbeck and Dan Freed on “Instantons and 4-Manifold Topology” instructed and inspired a generation of differential geometers. She continued to work in this area, and in particular had an important result with Lesley Sibner and Robert Sibner on non self-dual solutions to the Yang-Mills equations.

Integrable systems and harmonic mappings
The study of integrable systems has its roots in 19th century classical mechanics. Using the language of gauge theory, Uhlenbeck and Hitchin realised that harmonic mappings from surfaces to homogeneous spaces come in 1-dimensional parametrised families. Based on this observation, Uhlenbeck described algebraically harmonic mappings from spheres into Grassmannians relating them to an infinite dimensional integrable system and Virasoro actions. This seminal work led to a series of further foundational papers by Uhlenbeck and Chuu-Lian Terng on the subject and the creation of an active and fruitful school.

The impact of Uhlenbeck’s pivotal work goes beyond geometric analysis. A highly influential early article was devoted to the study of regularity theory of a system of non-linear elliptic equations, relevant to the study of the critical map of higher order energy functionals between Riemannian manifolds. This work extends previous results by Nash, De Giorgi and Moser on regularity of solutions of single non-linear equations to solutions of systems.

Karen Uhlenbeck’s pioneering results have had fundamental impact on contemporary analysis, geometry and mathematical physics, and her ideas and leadership have transformed the mathematical landscape as a whole.
In 1990, in Kyoto, Japan, Karen Uhlenbeck became only the second woman to give a Plenary Lecture at the International Congress of Mathematicians – ICM – the largest and most important gathering of mathematicians in the world. It is held every four years, and the first woman to do this was Emmy Noether in 1932. Such a shocking statistic reflects just how hard it is for many women to achieve the recognition they deserve in a male-dominated field.

But by that point in her career, Uhlenbeck had already established herself as one of the world’s preeminent mathematicians, having overcome many hurdles, both personally and professionally. In 2000, she received the US National Medal of Science. Yet for many, the recognition of her achievements should have been far greater, for her work has led to some of the most important advances in mathematics in the last 40 years.

Karen Keskulla Uhlenbeck, the eldest of four children, was born in Cleveland, Ohio in 1942. Her father, Arnold Keskulla, was an engineer and her mother, Carolyn Windeler Keskulla, an artist and school teacher. The family moved to New Jersey when Karen was in third grade. As a young girl, she was curious about everything. Her parents instilled in her a love of art and music, and she developed a lifelong love of the outdoors, regularly roaming the local countryside near her home.

Most of all, she loved reading, shutting herself away whenever she could to devour advanced science books, staying up late at night and even reading secretly in class. She dreamed of becoming a research scientist, particularly if it meant avoiding too much interaction with other people; not that she was a shy child, but rather because she enjoyed the peace and solitude of her own company. The last thing she wanted to do was to follow in her mother’s footsteps and end up teaching – an attitude that would change dramatically later in life.

Uhlenbeck’s love affair with mathematics developed only after she had
© Karen Uhlenbeck giving a talk at the Institute for Advanced Study. PHOTO: Andrea Kane
started at university. Having been inspired in high school by the writings of great physicists such as Fred Hoyle and George Gamow, she enrolled at the University of Michigan, initially planning to major in physics. However, she soon discovered that the intellectual challenge of pure mathematics was what really excited her. It also meant she didn’t have to do any lab work, which she disliked.

Graduating in 1964, she married her biophysicist boyfriend Olke Uhlenbeck a year later and decided to embark on postgraduate study. Already well aware of the predominantly male and often misogynistic culture in academia, she avoided applying to prestigious schools such as Harvard, where Olke was heading for his PhD and where competition to succeed was likely to be fierce. Instead, she enrolled at Brandeis University where she received a generous graduate fellowship from the National Science Foundation. There, she completed her PhD in mathematics, working on the calculus of variations; a technique that involves the study of how small changes in one quantity can help us find the maximum or minimum value of another quantity – like finding the shortest distance between two points. You might think this would be a straight line, but it is not always so straightforward. For example, if you have to drive through a busy city, the quickest route is not necessarily the shortest. Needless, to say, Uhlenbeck’s contribution to the field was somewhat more complicated than this!

After a brief teaching period at MIT, she moved to Berkeley, California, where she studied general relativity and the geometry of space-time – topics that would shape her future research work. Although a pure mathematician, Uhlenbeck has drawn inspiration for her work from theoretical physics and, in return, she has had a major influence in shaping it by developing ideas with a wide range of different applications.

For example, physicists had predicted the existence of mathematical objects called instantons, which describe the behaviour of surfaces in four-dimensional space-time. Uhlenbeck became one of the world’s leading experts in this field. The classic textbook, Instantons and 4-Manifolds, which she co-wrote in 1984 with Dan Freed, inspired a whole generation of mathematicians.

In 1971, she became an assistant professor at the University of Illinois at Urbana-Champaign where she felt isolated and undervalued. So, five years later she left for the University of Illinois at Chicago. Here, there were other female professors, who offered advice and support, as well as other mathematicians who took her work more seriously. In 1983, she took up a full professorship at the University of Chicago, establishing herself as one of the preeminent mathematicians of her generation. Her interests included nonlinear partial differential equations, differential geometry, gauge theory, topological quantum field theory and integrable systems. In 1987, she moved to the University of Texas at Austin to take up the Sid W. Richardson Foundation Regents’ Chair in mathematics. There, she broadened her understanding of physics
by studying with Nobel Prize winning
physicist Steven Weinberg. She would
remain at the University of Texas until the
end of her working career.

Uhlenbeck’s most noted work
focused on gauge theories. Her papers
analysed the Yang-Mills equations in four
dimensions, laying some of the analytical
groundwork for many of the most exciting
ideas in modern physics, from the
Standard Model of particle physics to the
search for a theory of quantum gravity.
Her papers also inspired mathematicians
Cliff Taubes and Simon Donaldson, paving
the way for the work that won Donaldson
the Fields Medal in 1986.

Uhlenbeck, now back in New Jersey,
remains a staunch advocate for greater
gender diversity in mathematics and in
science. She has come a long way from
the young girl who wished to be alone.
For a while, she struggled to come to
terms with her own success, but now
says she appreciates it as a privilege.
She has stated that she is aware of
being a role model, for young female
mathematicians in particular, but that
“it’s hard, because what you really need
to do is show students how imperfect
people can be and still succeed. Everyone
knows that if people are smart, funny,
pretty, or well-dressed they will succeed.
But it’s also possible to succeed with
all of your imperfections. I may be a
wonderful mathematician and famous
because of it, but I’m also very human.”
Karen Uhlenbeck is certainly a remarkable
human.
I’m forever blowing bubbles,
Pretty bubbles in the air.
They fly so high,
Nearly reach the sky,
Then like my dreams,
They fade and die.

Jaan Kenbrovin

Soap bubbles are beautiful objects, perfectly shaped and with a marvellous play of colours, due to interference of light reflecting off the front and back surfaces of the soap film. Soap bubbles are beautiful objects in a mathematical setting as well, as they constitute examples of minimal surfaces. When the enclosed volume of air inside the bubble is fixed, the soap film will minimize the wall tension, pulling the bubble into the shape of the least surface enclosing a fixed volume, known for centuries to be a perfect sphere.

If we instead of blowing the bubble, dip a heavily deformed wire loop into a soap bubble solution, the soap film will form a disc with its boundary given by the wire loop and of minimal area. Unlike the sphere-shaped bubble, this film has equal pressure on each side, hence it is a surface with zero mean curvature, i.e. the average curvature along all directions is zero. Even if the soap film almost instantly is able to form a minimal surface, computing the shape of the surface analytically is a rather complicated task.
Among curves connecting two points in space, we can always find a shortest path. The analogous statement is not true for surfaces when considering their area. The problem is that in order to reduce the area of a surface, a consequence could be that the surface is shrunk to a curve, which of course does not count as a minimal surface. An example of this is the minimal tubular surface connecting two parallel circles. If the distance between the circles is small compared to their radius, the minimal surface looks like a slightly concave cylinder. When pulling the circles apart the cylinder will shrink in the region between the circles, forming a surface known as a catenoid. At a certain point, the middle part of the curved cylinder will collapse along the line connecting the centres of the two parallel circles. When pulling the circles further apart there is no tubular minimal surface connecting them.

Mapping spaces
In 1968 Karen Uhlenbeck received her Ph.D. from Brandies with the thesis “The Calculus of Variations and Global Analysis”. Her supervisor was Richard Palais, who a few years earlier and together with Stephen Smale, had introduced the so-called Palais-Smale Condition C. This condition gives a criterion for the existence of minimizers for functionals on mapping spaces. “Minimizers for functionals on mapping spaces” is a more general phrase than “find the surface of least area”, but Condition C can also be applied to the minimal surface problem, but then it fails. Motivated by the general non-existence of minimal surfaces, Uhlenbeck wanted to understand what happens when Condition C is violated. In a paper co-authored with Jonathan Sacks, they describe in detail the situation where you cannot rely on the conclusion of Condition C. They construct a sequence of mappings of a sphere into the target space which satisfies Condition C, but in such a way that their limit does not. Outside of a finite set of singular points everything works well, but near the singularities the so-called bubbling phenomenon appears. The area of the limit surface is strictly less than the limit of the areas of the surfaces in the sequence. The difference is concentrated in a finite set of isolated points, being the limit of “bubbles” in the sequence of surfaces. The idea and the methods of this revolutionary paper has since it was published become a successful mathematical tool. In particular, the bubbling phenomenon has had great influence as a method for solving problems in various parts of mathematics.

Footprints of gauge
Karen Uhlenbeck also left her footprints in the field called gauge theory. Gauge theory is a mathematical theory introduced by Hermann Weyl in 1918, which originated in theoretical physics and Einstein’s theory of general relativity. A key idea in Einstein’s work is that laws of physics should be the same in all frames of reference. This is also the general idea of a gauge theory, to find connections that compare measurements taken at different points in a space and look for quantities that do not change. The physical
interpretation was brought further by Yang and Mills in the fifties, in what is now called the Yang-Mills equations. To reveal the secrets of theoretical physics you have to work in a (at least) four-dimensional space, three spatial coordinates and one time-coordinate. A physical law should be the same wherever you are located in space-time, i.e. independent of choice of frame of reference.

**Minimal surfaces**
Karen Uhlenbeck attacked this problem from the mathematical point of view. She pioneered the study of Yang-Mills equations in a rigorous analytical way. Her work formed a base of all subsequent research in the area of gauge theory. Her analysis of the Yang-Mills equations in four dimensions together with C. H. Taubes, also laid the ground for the theories of Simon Donaldson, who later was awarded the Fields Medal in 1986 for his work on the topology of four-manifolds.

Minimal surfaces and gauge theory are two separate fields of mathematics which both originate from a wish to understand nature. When mathematicians get interested in such problems, the theory propagates into theoretical constructions far beyond the tangible objects of nature. But even if the mathematical theory seems to be soaring, scientists often benefit from the generalized theory. Karen Uhlenbecks mathematical achievements constitute important examples of such processes.

*(I’m Forever Blowing Bubbles is an American song from the Broadway musical* The Passing Show of 1918. *It was released in 1918, the same year as Hermann Weyl introduced the notion of a gauge. In addition to the obvious connection to minimal surfaces, the lyrics may have some associations to mathematical research in general. The song is also adapted as the club anthem of West Ham United, a London-based football club.)*
Karl Johans gate, the main street of Oslo during the Abel week.

PHOTO: Thomas Brun/NTB
The Abel Prize is an international award for outstanding scientific work in the field of mathematics, including mathematical aspects of computer science, mathematical physics, probability, numerical analysis, scientific computing, statistics, and also applications of mathematics in the sciences.

The Abel Prize has been awarded since 2003 by the Norwegian Academy of Science and Letters. The choice of laureates is based on the recommendations from the Abel Committee. The prize carries a cash award of 6 million NOK (about 650,000 Euro or about 730,000 USD).

The prize is named after the exceptional Norwegian mathematician Niels Henrik Abel (1802–1829). According to the statutes of the Abel Prize the objective is both to award the annual Abel Prize, and to contribute towards raising the status of mathematics in society and stimulating the interest of children and young people in mathematics.

Among initiatives supported are the Abel Symposium, the International Mathematical Union’s Commission for Developing Countries, and the Bernt Michael Holmboe Memorial Prize for excellence in teaching mathematics in Norway. In addition, national mathematical contests, and various other projects and activities are supported in order to stimulate interest in mathematics among children and youth.

At the Heidelberg Laureate Forum in Germany young mathematicians get the opportunity to meet winners of the Abel Prize.

**Call for nominations 2020**

The Norwegian Academy of Science and Letters hereby calls for nominations for the Abel Prize 2020, and invites you (or your society or institution) to nominate candidate(s). Nominations are confidential and a nomination should not be made known to the nominee.

Deadline for nominations for the Abel Prize 2020 is September 15, 2019.

Please consult [www.abelprize.no](http://www.abelprize.no) for more information.
The laureate wall in The Abel-room at the Norwegian Academy of Science and Letters. PHOTO: Eirik Furu Baardsen
2018 Robert P. Langlands
“for his visionary program connecting representation theory to number theory”

2017 Yves Meyer
“for his pivotal role in the development of the mathematical theory of wavelets.”

2016 Sir Andrew J. Wiles
“for his stunning proof of Fermat’s Last Theorem by way of the modularity conjecture for semistable elliptic curves, opening a new era in number theory.”

2015 John Forbes Nash, Jr. and Louis Nirenberg
“for striking and seminal contributions to the theory of nonlinear partial differential equations and its applications to geometric analysis.”

2014 Yakov G. Sinai
“for his fundamental contributions to dynamical systems, ergodic theory, and mathematical physics.”

2013 Pierre Deligne
“for seminal contributions to algebraic geometry and for their transformative impact on number theory, representation theory, and related fields.”

2012 Endre Szemerédi
“for his fundamental contributions to discrete mathematics and theoretical computer science, and in recognition of the profound and lasting impact of these contributions on additive number theory and ergodic theory.”

2011 John Milnor
“for pioneering discoveries in topology, geometry and algebra.”
2010  John Torrence Tate
“for his vast and lasting impact on the theory of numbers.”

2009  Mikhail Leonidovich Gromov
“for his revolutionary contributions to geometry.”

2008  John Griggs Thompson
and Jacques Tits
“for their profound achievements in algebra and in particular
for shaping modern group theory.”

2007  Srinivasa S. R. Varadhan
“for his fundamental contributions to probability theory and
in particular for creating a unified theory of large deviations.”

2006  Lennart Carleson
“for his profound and seminal contributions to harmonic
analysis and the theory of
smooth dynamical systems.”

2005  Peter D. Lax
“for his groundbreaking contributions to the theory and application of
partial differential equations and to the computation of their solutions.”

2004  Sir Michael Francis Atiyah
and Isadore M. Singer
“for their discovery and proof of the index theorem, bringing together
topology, geometry and analysis, and their outstanding role in building
new bridges between mathematics and theoretical physics.”

2003  Jean-Pierre Serre
“for playing a key role in shaping the modern form of many
parts of mathematics, including topology, algebraic geometry
and number theory.”
After hearing a talk by 2004 Abel laureate, the late Sir Michael Atiyah, Karen Uhlenbeck became interested in gauge theory. PHOTO: Didier Vandenbosch
Programme
Abel Week 2019

May 20

Holmboe Prize Award Ceremony
Jan Tore Sanner, Minister of Education and Integration, presents the Bernt Michael Holmboe Memorial Prize for teachers of mathematics at Oslo Cathedral School

Wreath-laying at the Abel Monument
by the Abel Prize Laureate in the Palace Park

Dinner in honour of the Abel Laureate
at the Norwegian Academy of Science and Letters (by invitation only)

May 21

Abel Prize Award Ceremony
His Majesty King Harald V presents the Abel Prize to the Laureate in the University Aula, Oslo, Norway

Reception and interview with
the Abel Laureate
at Det Norske Teatret, Oslo, Norway

May 22

The Abel Lectures
The Laureate will give the Abel Prize lecture at the University of Oslo, Georg Sverdrups Hus, Aud. 1. This will be followed by other lectures with topics related to the prize winner’s work

The Abel Party
at the Norwegian Academy of Science and Letters (by invitation only)

May 23

Abel Prize Lecture and mathematical games
The Abel Laureate gives a lecture at the University of Bergen. Schoolchildren will invited to play mathematical games with the prize winner in Archimedes Labyrinth in Bergen Botanical Garden

Register online at: www.abelprize.no from mid-April or contact abelprisen@dnva.no, facebook.com/Abelprize

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